ESS515/415 E&M Review

Maxwell's Equations (MKS)

$$\vec{\nabla} \times \vec{\mathbf{E}} = -\frac{\delta \vec{\mathbf{B}}}{\delta \mathbf{t}}$$ Faraday's Law

$$\vec{\nabla} \times \vec{\mathbf{B}} = \mu_{\mathbf{o}} \vec{\mathbf{J}} + \mu_{\mathbf{o}} \varepsilon_{\mathbf{o}} \frac{\delta \vec{\mathbf{E}}}{\delta \mathbf{t}}$$ Ampere's Law

$$\vec{\nabla} \cdot \vec{\mathbf{B}} = 0$$ No Magnetic Monopoles

$$\vec{\nabla} \cdot \vec{\mathbf{E}} = \frac{\rho_{\mathbf{c}}}{\varepsilon_{\mathbf{o}}}$$ Gauss' Law

Source Terms:

$$\rho_{\mathbf{c}} = \textbf{ChargeDensity} = \frac{\sum_{\mathbf{k}=1}^{\mathbf{N}} \mathbf{q}_{\mathbf{k}}}{\Delta \mathbf{V}} = \mathbf{e}(\mathbf{n}_{+} - \mathbf{n}_{-}) 2 \text{ species}$$

(N is number of charges in $\Delta \mathbf{V}$ volume)

$$\vec{\mathbf{J}} = \text{ Current Density } = \frac{\sum_{\mathbf{k}=1}^{\mathbf{N}} \mathbf{q}_{\mathbf{k}} \vec{\mathbf{v}}_{\mathbf{k}}}{\Delta \mathbf{V}} = \sum_{\mathbf{s}} \mathbf{n}_{\mathbf{s}} \, \mathbf{e}_{\mathbf{s}} \, \vec{\mathbf{v}}_{\mathbf{s}}$$

References:

Parks: Chapter 2

Jackson: Classical Electrodynamics

Lorraine and Corson: Electromagnetic Fields and Waves

$\vec{B} = m_o \vec{H}$ in plasma $m \rightarrow m_o$ = permeability of free space

$\vec{D} = e_o \vec{E}$ where $e \Rightarrow e_o$ = permitivity of free space

$m_o \cong 4p \times 10^{7} \cong 1.26 \times 10^{-6}$ henry/m

$e_o \cong 8.85 \times 10^{-12}$ farad/m

Speed of Light $c = 1/\sqrt{m_o e_o}$ in free space

Lorentz force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ where $q = Ze$

Coulomb's Law

$$\vec{E}(\vec{r}) = \frac{1}{4pe_o} \sum_i q_i \frac{\vec{r} - \vec{r}_i}{\left|\vec{r} - \vec{r}_i\right|^3} \quad or \quad \vec{E}(\vec{r}) = \frac{1}{4pe_o} \int_V \frac{\vec{r} - \vec{r}\,'}{\left|\vec{r} - \vec{r}\,'\right|^3} r(r')d^3 r$$

Sum over point charges — integral over volume charge density

Biot-Savart Law

$$\vec{B}(\vec{r}) = \frac{m_o}{4p} I_1 \oint \frac{d\vec{l}_1 \times \left|\vec{r} - \vec{r}_1\right|}{\left|\vec{r} - \vec{r}_1\right|^3}$$ Field at $\vec{r}$ due to current loop I_1

Note both $\vec{E}$ *and* $\vec{B}$ *have* $\frac{1}{r^2}$ dependence to elemental current $d\vec{l}$ or elemental charge $dr = r d^3 r$

Lorentz force on a point charge $\rightarrow$Force/volume

for charge density $r_c = e(n_+ - n_-)$ for 2 species, or

$$r_c = \sum_s Z_s e_s n_s \quad \text{for s species}$$

(where s=electrons, ion1, ion2, etc, Z is net ion charge, e is + or - one unit of charge)

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B}) \Rightarrow \vec{F}/vol = r_c(\vec{E} + \vec{v}\times\vec{B}) = r_c\vec{E} + r_c\vec{v}\times\vec{B}$$

$$= r_c\vec{E} + ne\vec{v}\times\vec{B} \text{ (for single species, or 1-fluid theory)}$$

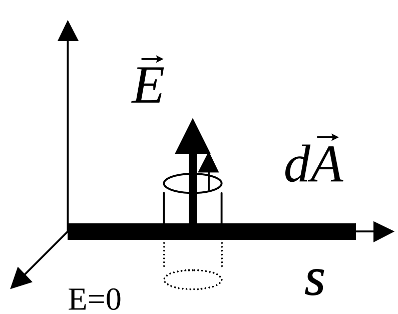
In general: $\boxed{\vec{F}/vol = r_c\vec{E} + \vec{j}\times\vec{B}}$

Divergence Theorem $\int \nabla\cdot\vec{A}d^3r = \oint \vec{A}\cdot\hat{n}ds$

Example: $\vec{E}$ field next to a charged conductor with surface charge s :

Use Gauss' law $\int \nabla\cdot\vec{E}d^3r = \int \frac{rd^3r}{e_o} = \int \frac{sd^2r}{e_o}$

$$\oint \vec{E}\cdot\hat{n}ds = \vec{E}\cdot\vec{A} = \frac{ó}{å_o}A \;\; \therefore \;\; \vec{E} = \frac{ó}{å_o}\hat{n} \text{ (normal to surface A)}$$



where $\vec{A} = \hat{n}A$

Steady State Solutions for $\vec{E}$

Faraday: $\nabla \times \vec{E} = -\partial \vec{B} / \partial t \approx 0 \equiv$ steady state

$\Rightarrow \vec{E}$ is derivable from a potential since

$\nabla \times \nabla \vec{A} \equiv 0$

So, let $\boxed{\vec{E} = -\nabla V}$ where V = potential (scalar field),

From Gauss' Law $\nabla \cdot \vec{E} = \frac{}{e_o} \Rightarrow \boxed{\nabla^2 V = -\frac{r}{e_o}}$

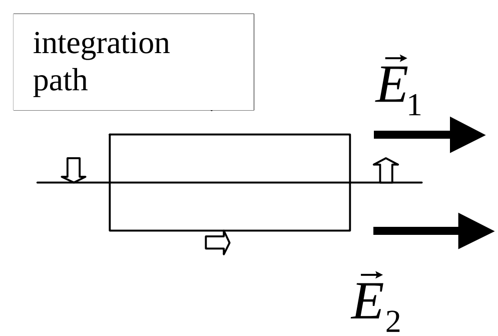
Poisson's Equation

E&M Boundary Conditions:

In steady state $\nabla \times \vec{E} = 0$ so we can use

$$(\vec{\ } \times A) \cdot \hat{n} dS \quad \oint \vec{A} \cdot d\vec{l}$$

$$\int (\vec{\nabla} \times \vec{E}) \cdot n dA = \oint \vec{E} \cdot d\vec{l}$$



$$\vec{E}_1 \cdot d\vec{l} \quad E_2 \cdot d\vec{l} =$$

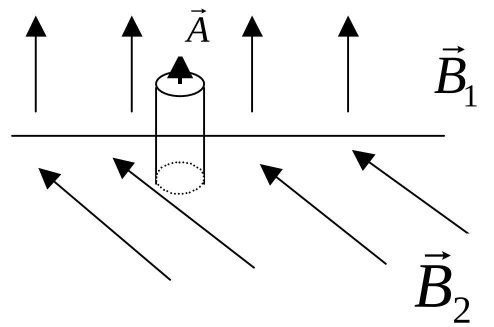
So ${}_{1t} = E_{2t}$ Tangential Component of $\vec{E}$ is continuous across a boundary in the steady state

Magnetic Field at boundary

$$\nabla \cdot \vec{B} = 0 = \oint \vec{B} \cdot d\vec{A}$$

$$-\vec{B}_1 \cdot \vec{A} + \vec{B}_2 \cdot \vec{A} = 0$$

or



$$B_{1n} = B_{2n}$$

Normal Component of $\vec{B}$ Continuous at an interface

Conservation of charge

Look at $\nabla \times \vec{B} = m_o \vec{J} + m_o e_o \dfrac{\partial \vec{E}}{\partial t}$

take the divergence, remembering that $\nabla \cdot (\nabla \times \vec{A}) \equiv 0$

0= $m_o \nabla \cdot \vec{J} + m_o e_o \dfrac{\partial \nabla \cdot \vec{E}}{\partial t}$, but $\nabla \cdot \vec{E} = \dfrac{r}{e_o}$, so

$\boxed{0 = \nabla \cdot \vec{J} + \dfrac{\partial r}{\partial t}}$ Conservation of Charge

Note that in the steady state $\nabla \cdot \vec{J} = 0$ which is analogous to Kirchoff's Law $\sum I_j = 0$

Ohm's law

Simple form: $\vec{J} = s\,\vec{E}$ where s is the conductivity

In the steady state $\nabla \cdot J = 0 \quad \Rightarrow \quad \nabla \cdot (s\,\vec{E}) = 0$

so $\nabla s \cdot \vec{E} + s\,(\nabla \cdot \vec{E}) = 0$

gradient of conductivity $\Uparrow$ $\qquad\qquad$ $\Uparrow = r / e_o$

Use this to determine $\vec{E}$ given s and r

s (conductivity) depends on the medium properties.
In a plasma s is often very large $s \Rightarrow \infty$. (We will derive s for a plasma)

Energy density

Total Electric energy $W = \frac{1}{2}\int r V d^3 r$ (the factor of 2 is to avoid counting charges twice, See L&C p. 72)

From Poisson's Equation $r = -e_o \nabla^2 V$

$W = -e_o / 2 \int V \nabla^2 V d^3 r$ and use the vector identity:

$\vec{\nabla} \cdot (f\vec{A}) = f(\nabla \cdot A) + \vec{A} \cdot \nabla f$ along with the Divergence theorem to get

$$W = \frac{e_o}{2}\int E^2 d^3 r$$

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Energy Density (continued)

so $e_E = \dfrac{e_o E^2}{2}$ is the electric field energy density

Similarly for B-field

$$e_B = \frac{B^2}{2m_o}$$

So we have $e_{EM} = \dfrac{e_o E^2}{2} + \dfrac{B^2}{2m_o}$

Electromagnetic energy Density

Lenz' Law

$$\int \nabla \times \vec{E} \cdot d\vec{A} = -\frac{\partial}{\partial t} \int \vec{B} \cdot dA$$

from Stoke's Theorem:

$\oint \vec{E} \cdot dl = -\dfrac{\partial}{\partial t}\Phi$ were Φ is the magnetic flux through the surface and the left hand side is the E.M.F.

$\vec{J} = s\vec{E}$ so $\dfrac{1}{s}\oint J \cdot d\vec{l} = -\dfrac{\partial}{\partial t}\Phi$

Current flows in the loop to balance any change in magnetic flux through the loop. This is very useful in plasma physics from microscopic to macroscopic dimensions.