

III Longitudinal Drift

for $J = 0 = \nabla \times B$

$$V_G = \frac{\overline{B} \times \nabla \overline{B}}{eB^3} (\epsilon_{\perp} + 2\epsilon_{\parallel})$$

integrate around drift-path of 360° Longitude

gives $\tau_{\text{drift}} \approx \frac{44}{r_e * \epsilon \text{ (mev)}} \text{ minutes}$
 $r_e \leftarrow \text{in earth radii}$

That is, $r_e \equiv L R_e \equiv L$ earth radii

for $\epsilon = 0.1 \text{ Mev}$ and $r_e = 10 R_e$

$$\tau_{\text{drift}} \approx 44 \text{ minutes}$$

so, in general

$$\tau_{\text{drift}} \gg \tau_{\text{bounce}} \gg \tau_{\text{gyro}}$$

Neglecting scattering and plasma instabilities
particles can be trapped for ever

in practice some species at some energies
are trapped for 100 yrs!

Radiation Belt organization

For a dipole B -field with
 M = dipole moment of the earth

$$\left[\vec{M}_{\text{dipole}} \oint \vec{u} \cdot d\vec{l} (\vec{r}, \vec{u}, t) d^3v \right] = \text{magnetic moment/volume}$$

≡ Magnetization

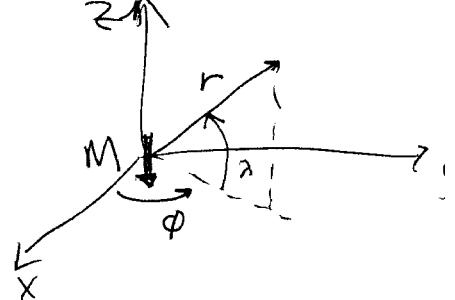
For a dipole $\vec{B} = -\vec{\nabla} \psi$ (remind of
(Panels (3.10)))

where $\psi = -\frac{1}{r} \vec{M} \cdot \vec{\nabla} \left(\frac{1}{r} \right)$ for dipole

in spherical coords:

$$B_r = \frac{\partial}{\partial r} \psi = -\frac{1}{r^2} M \frac{\sin \lambda}{r^3}$$

$$B_\theta = \frac{1}{r \cos \lambda} \frac{\partial}{\partial \phi} \psi = 0$$



$$B_\lambda = -\frac{1}{r} \frac{\partial}{\partial \lambda} \psi = M \frac{\cos \lambda}{r^3}$$

$$|\vec{B}| = \sqrt{B_r^2 + B_\lambda^2 + B_\phi^2} = \frac{M}{r^3} (1 + 3 \sin^2 \lambda)^{1/2}$$

~~at~~

from Panels 3.30 / 3.31 can write

$$\frac{dr}{r} = -\frac{d(\cos \lambda)}{\cos \lambda}$$

integrate to get

$$\phi = \phi_0 \quad \underline{r = r_0 \cos^2 \lambda}$$

Now let $L \equiv \frac{r_0}{R_e}$ marks Equatorial distance
of a particular Field Line

$$\text{so } \underline{r = L R_e \cos^2 \lambda}$$

over →

or, to put it another way,

The Latitude at the surface is given by $r = R_e$ so $\cos^2 \lambda = \frac{1}{L}$

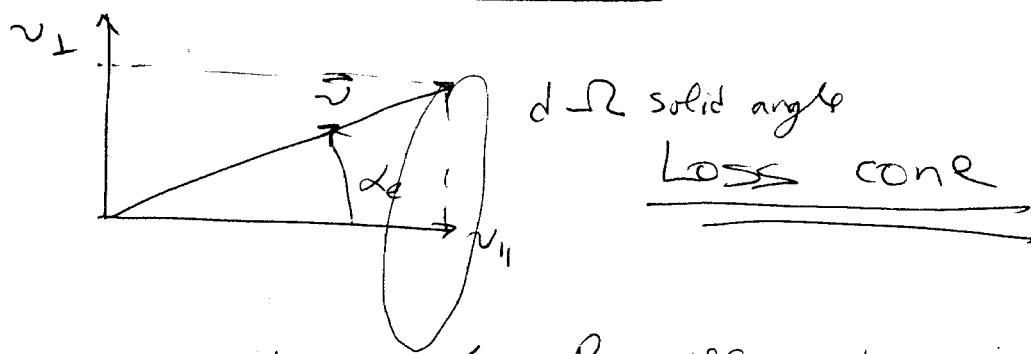
This is definition of Invariant Latitude

L can be defined more carefully for a distorted field.

So, for a given L shell most particles will be undergoing 3 distinct motions - (gyration, bounce & VB drift) but if mirror point is $< 1 R_e$ they will be lost due to scattering from atmosphere.

Actually, the loss altitude $\sim R_e + 100\text{ km}$ where probability of scattering becomes high

Which Particles are Lost?



If mirror point is $< R_e + 100\text{ km}$, then find this $\alpha_e \rightarrow$
particle is lost.

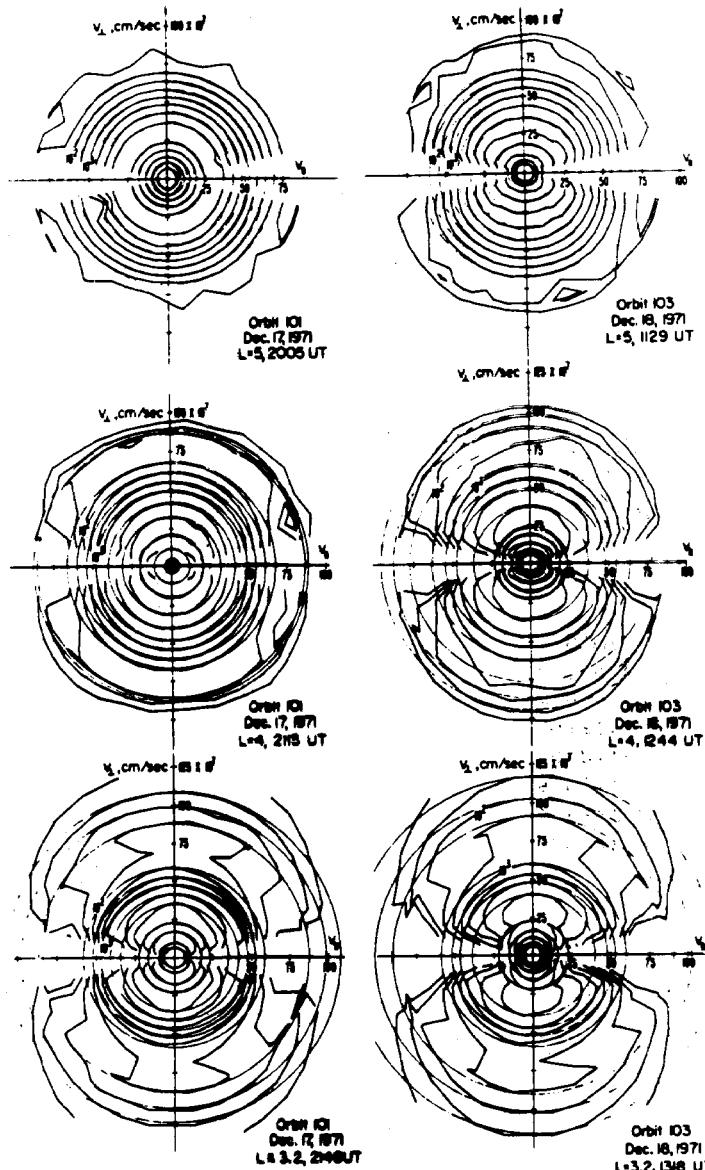


Figure 7. Contours of constant phase space density, $f = j/E$, plotted in v_{\perp} , v_{\parallel} plane. Circles centered at $v_{\perp} = v_{\parallel} = 0$ (thin lines) added for reference. Contours generated every 0.5 units in $\log_{10} f$ by computer with no smoothing applied to the data. Contours generated for $v_{\perp} > 0$ and $v_{\parallel} < 0$ separately and region of no available data separates the two sets of contours. Spatial evolution of distribution function shown during main phase (orbit 101) and recovery phase (orbit 103). Count rate statistics are the cause of jagged nature of several contours. Break in orbit 101, $L = 5$ contours is a telemetry problem.

Figure 8.
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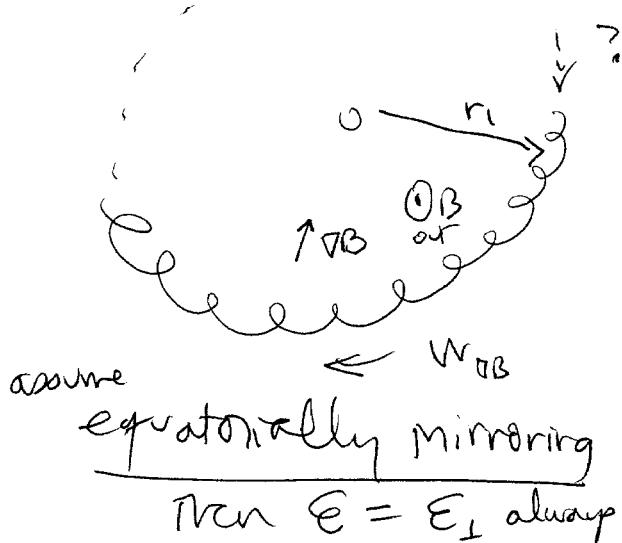
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$$\text{using } \mu = \text{constant} \sim \frac{v^2 \sin^2 \alpha}{B_e}$$

find α_e such that $\alpha = 90^\circ$ at B for $r = R_E + 100 \text{ km}$
 we find that $\alpha_e \sim 3^\circ$ at $L = 6$
 so MOST particles are trapped.

Show Pitch angle Distributions along Field Line

L-Shell drift: How can you tell
 the drift motion always returns guiding
 center to starting point?



Answer: If energy is
 conserved then
 with $\mu = \text{constant}$, if
 start from r_1 where $B = B_1$,

$$\mu = \frac{\epsilon}{B_1} \text{ at } r_1$$

If returned at $r \neq r_1$, $B \neq$
 $\Rightarrow \mu \neq \text{constant}$

L defines a closed shell for perfect dipole

Adiabatic Invariants

We derived the First Invariant M directly from equations of motion. $M = \text{constant}$ as long as the magnetic field varies only slightly from a uniform field in which the particle gyrates periodically. With these two conditions Slow variation and periodic motion

adiabatic invariants can be derived generally.

Assume motion described by Hamilton's Equations

$$\frac{dp_i}{dt} \equiv \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \frac{dq_i}{dt} \equiv \dot{q}_i = \frac{\partial H}{\partial p_i} \quad i = 1, 2, 3$$

Hamiltonian differs only slightly from one describing periodic motion

i.e. actual $H = H(\bar{p}, \bar{q}, \delta)$ where ~~δ~~ δ is small parameter eg $\delta \rightarrow \frac{P}{L}$

with no perturbation ($\delta = 0$) eq. uniform field motion is simply periodic - i.e. all components periodic with same period.

We can show that there exist variables \vec{z}, θ such that

transformations

$$\begin{aligned}\vec{P} &= \vec{P}(\vec{z}, \theta, \delta) \\ \vec{q} &= \vec{q}(\vec{z}, \theta, \delta)\end{aligned}\left.\right\} \begin{array}{l} \text{periodic} \\ \text{in } \theta \end{array}$$

so that

$$\frac{d\vec{z}}{dt} = \delta \vec{h}(\vec{z}, \delta)$$

$$\frac{d\theta}{dt} = \omega(\vec{z}, \delta)$$

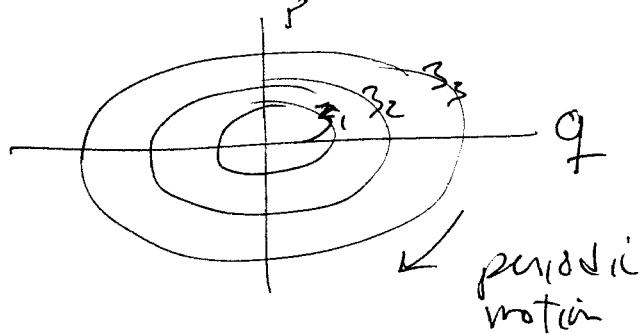
(all to 1st order in δ) (Proof: see Northrup
- ref. in Parks)

Physical Significance ($\delta = 0$ case)

$$\frac{d\vec{z}}{dt} = 0 \quad \therefore \quad \vec{z} = \text{constant}$$

$$\text{and } \frac{d\theta}{dt} = \omega(\vec{z}, 0) = \text{const}$$

Phase space orbits - closed surfaces



\vec{z} is constant for each orbit
(e.g. for 1-dim harmonic oscillator -
 \vec{z} of total energy)

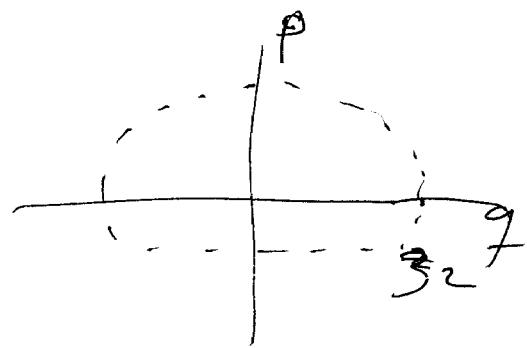
θ increases linearly with t
(\vec{z}, θ are related to action-angle variables)

as motion is perturbed \vec{z} no longer constant
but a) \vec{z} varies slowly

b) change in \vec{z} independent of θ

so, all particles starting initially with
same $\vec{z} = \vec{z}_1$ later all have same

$$\vec{z} = \vec{z}_2 + \vec{z}_1$$



Adiabatic Invariant

$$I(\vec{z}, \delta) = \oint d\theta \vec{p}(\vec{z}, \theta, \delta) \cdot \frac{\partial \vec{q}(\vec{z}, \theta, \delta)}{\partial \theta}$$

Usually abbreviated

$$I = \oint_{\vec{z}=\text{constant}} \vec{p} \cdot d\vec{q} = \int d^3 p d^3 q$$

but $\int d^3 p d^3 q$ has same value for any canonical \vec{p}, \vec{q} :

\Rightarrow since transformation from unperturbed to perturbed \vec{p}, \vec{q} is canonical

$$\int_{\text{actual}} d^3 p d^3 q = \int_{\text{unperturbed}} d^3 p d^3 q = \int_{\text{unperturbed mot.}} \vec{p} \cdot d\vec{q}$$

I^{st} Invariant

$$H = \sqrt{(\vec{p} - \frac{e}{c} \vec{A})^2 c^2 + m^2 c^2} + e \vec{\Phi}$$

$$\vec{\Pi} = \vec{p} + \frac{e}{c} \vec{A} \quad \vec{p} \text{ canonical momentum}$$

$\sqrt{\quad} = \text{relativistic momentum}$

can show $I_1 = \pm \frac{e p_{\perp} v_{\perp}}{B} = \gamma \mu$

$$I_2 = \oint dl p_{\parallel} \equiv j \quad \text{bounce motion}$$

$$j = p \oint dl \sqrt{1 - \frac{B}{B_m}} \quad \frac{j^2}{B_m} = \frac{p_{\perp}^2}{B}$$

$$I_3 = \oint \vec{\Pi} \cdot dl \quad \vec{\Pi} = \vec{p} + \frac{e}{c} \vec{A} \quad \begin{matrix} \text{canonical} \\ \text{vector potential} \end{matrix}$$

int path

$$= \oint \frac{e}{c} \vec{A} \cdot dl$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$= \frac{e}{c} \underbrace{\int \vec{B} \cdot d\vec{A}}_{\text{flux of } B \text{ field through surface}}$$

Stokes theorem
flux of B field through surfaces
defined by drift path

$$= \frac{e \vec{\Phi}}{c}$$

so $\vec{\Phi} = \frac{\text{flux through}}{\text{drift path is constant}}$

Lecture 12

$e^- + p$ - period vs L Plot

3rd Adiabatic invariant

$$\vec{\Pi} = \vec{p} + \frac{e}{c} \vec{A} \quad \text{canonical momentum}$$

$$\oint_{\text{drift path}} \vec{\Pi} \cdot d\vec{l}$$

1st term is of order e^2
 (Gyro and bounce motion
 are averaged over on time
 scales of the drift period)

$$\oint \frac{e}{c} \vec{A} \cdot d\vec{l} \quad \text{but } \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\text{so } \frac{e}{c} \underbrace{\int \vec{B} \cdot d\vec{s}}_{\text{flux of magnetic field through surface } S} \quad (\text{Stokes theorem})$$

flux of magnetic field through surface S
 defined by drift path

$$\text{so } I_B = \frac{e \Phi}{c}$$

Φ = flux through path
 again, like Lenz law -
 flux through conduct =
 constant.