

$$\text{Last time } \langle V_{G\parallel} \rangle = e E_{\parallel\parallel} - \mu \frac{\partial}{\partial \ell} \vec{B}$$

$$\frac{\partial}{\partial \ell} \equiv \frac{\vec{B}}{B} \cdot \vec{\nabla}$$

mentioned that ∇B and curvature drift can be combined when $\bar{J} = \bar{J} = \bar{\nabla} \times \bar{B}$ (vacuum)

$$\begin{aligned} \text{Grad. + Curv. Drift} &= \frac{e}{B^3} (\bar{B} \times \bar{\nabla} B) \left(\frac{mv_\perp^2}{z} + mv_{\parallel\parallel}^2 \right) \\ \bar{J} &= 0 \\ &= \frac{e}{B^3} (\bar{B} \times \bar{\nabla} B) (K_\perp + 2K_{\parallel\parallel}) \end{aligned}$$

where $K_\perp \equiv \frac{1}{2}mv_\perp^2$ $K_{\parallel\parallel} \equiv \frac{1}{2}mv_{\parallel\parallel}^2$

Now, look at current density

$$\bar{J} \equiv \sum_s n_s e_s v_s \quad s = \text{species}$$

for a particular species with charge e and energy K - Can we get J from the ∇B + curv. drifts for $V_s = \langle V_g \rangle$?

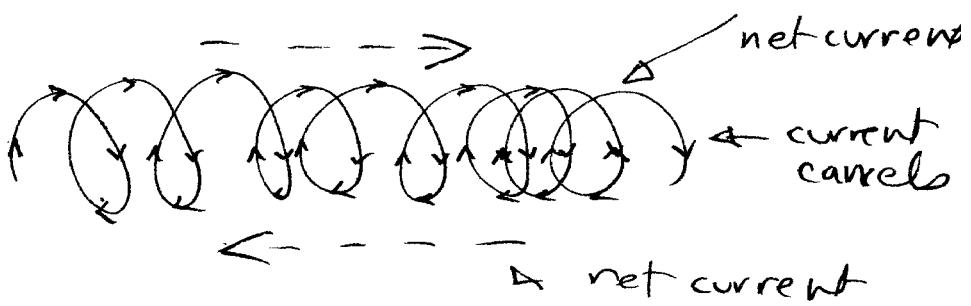
NO! - NOT reliable

Guiding center currents are only part of the total current

the particle gyro motion itself produces additional current if there is a gradient in the magnetic moment density. This current is analogous to the magnetization current in solids ($\nabla \times \text{Magnetization}$).

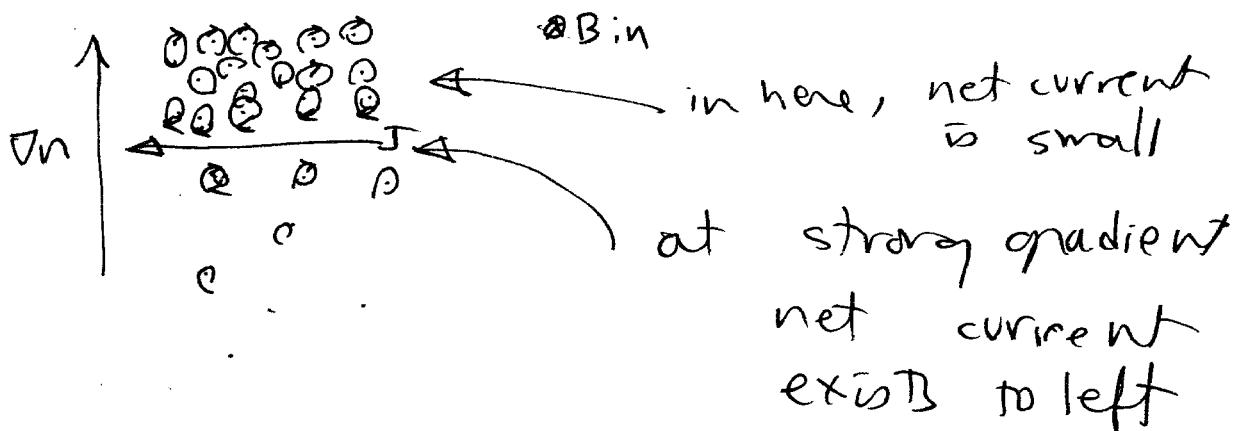
consider

e^+



so, the actual currents may have little to do with the value of $\nabla \times \vec{B}$, etc.

another way to look at this is with
a ∇n



Actually, the sign of the net current can often be determined from the gradient drifts, but not the magnitude of \vec{j}

→ (as discussed previously)

Meaning of μ = Particle magnetic moment

For a current loop, magnetic moment = current \times Area

Show that μ is a magnetic moment

current = $e \times (\# \text{ of times particle gyrates in } 1 \text{ sec})$

$$= \frac{e}{T_{\text{gyration}}} \quad \text{where } V_{\perp} T_g \equiv 2\pi\rho$$

$$\text{Area} = \pi \rho^2$$

$$\rho = \frac{mv_{\perp}}{1eIB}$$

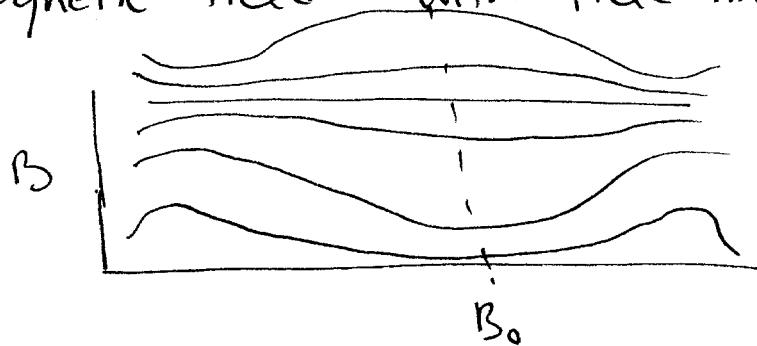
$$\therefore \text{magnetic moment} = \frac{eV_{\perp}}{2\pi\rho} \times \pi\rho^2 = \frac{eV_{\perp}\rho}{2}$$

$$\boxed{\mu = \frac{mv_{\perp}^2}{2B}} = \frac{k_{\perp}}{B}$$

Applications of Guiding Center Equations

Magnetic Bottle

Consider a cylindrically symmetric magnetic field with field lines as

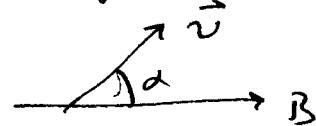


Suppose you start a particle on the axis at the place where field strength is B_0 .

Suppose the velocity of the particle makes an angle α with respect to the magnetic field.

What Happens?

α = pitch angle



$$v_{||} = v \cos \alpha$$

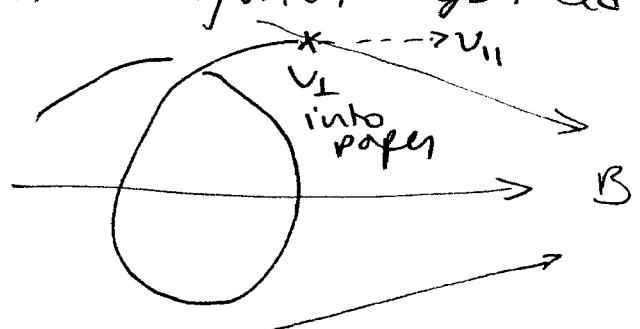
$$v_{\perp} = v \sin \alpha$$

$v_{||}$ carries the particle into a region of larger field strength some time

Later we have

$v_{||}$ to right

v_{\perp} into paper



at the particle \vec{B} has a component B_1 , that is parallel to the guiding center field and a component B_2 that is perpendicular.

So we have

$$\textcircled{X} \rightarrow v_{\parallel}^{\prime \prime}$$



consider components of $\vec{v} \times \vec{B}$ (Lorentz Force)

$v_{\parallel} B_2$ causes force that increases v_{\perp}
 $v_{\perp} B_2$ " " " decreases v_{\parallel}

thus, v_{\parallel} decreases while v_{\perp} increases.

Since $v_{\parallel}^2 + v_{\perp}^2 = \text{constant}$ (energy conserved),
after awhile v_{\parallel} will be zero. But
 $v_{\perp} B_2$ continues to act, so v_{\parallel} changes
sign

i.e.: Particle enters magnetic field a
certain distance, stops, turns around
and comes out!

This is called Mirroring

Since this happens at both ends of
the geometry, a particle may be trapped
and bounce back and forth between
end points indefinitely.

Proof of Same Result from Guiding Center Equations
we had $\frac{d}{dt} mv_{\parallel} = e E_{\parallel} - M \frac{\partial}{\partial v} B$

If $\frac{\partial}{\partial v} B > 0$ (ie field strength increasing along trajectory)

Then $\frac{d}{dt} mv_{\parallel} < 0$ and v_{\parallel} decreases

We want to know the point where the particle turns around, or the Mirror Point

Equation for v_+ is $\mu = \text{constant}$
 $\text{or } \frac{\frac{1}{2}mv_+^2}{B} = \text{const.}$

If particle moves into region where B is larger, then its v_+ must get larger too.

Question? How far into the region of increasing magnetic field will a particle penetrate if it starts with pitch angle α_0 at a place where $B = B_0$?

$$\mu = \frac{\frac{1}{2}mv_+^2 \sin^2 \alpha_0}{B_0} = \frac{\frac{1}{2}mv^2 \sin^2 \alpha}{B} \quad \left(v_+ = v \sin \alpha \text{ initially} \right)$$

as it penetrates, B increases, so does $\sin \alpha$ until, at the mirror point $\alpha = 90^\circ$

$$\text{and } v_{||} = v \cos \alpha = 0$$

$$v_+ = v \sin \alpha = v \leftarrow \text{all energy is lost}$$

Define B at this point to be B_m

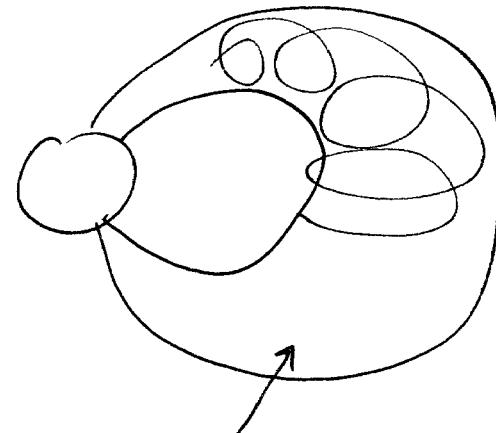
$$\text{we have } \frac{\sin^2 \alpha_0}{B_0} = \frac{1}{B_m}$$

$$\text{or } B_m = \frac{B_0}{\sin^2 \alpha_0}$$

All particles of any energy, charge, mass will mirror at same point B_m if they start with same α_0 !

in principle, if there are no collisions, and P/L is small enough, particles will be trapped forever (Goal of magnetic fusion)

Another Example of Guiding Center Motion
Particles in a Magnetic Dipole Field
The Radiation Belt



Bounce motion

as v_{\parallel} carries particles away from the equator they enter region of increasing B field
 \therefore mirror and bounce back & forth

Additionally, the guiding center drifts around the earth in longitude due to gradient + curvature drift.

I Gyration

$$\text{Gyration} = \text{gyroperiod} = \frac{2\pi}{\omega} = \frac{2\pi m}{Be} \sim 10^{-5} \text{ s } e^- \text{ at low } r = 10 R_E$$

$10^{-2} \text{ sec } p \text{ at } r = 10 R_E$

$10^{-2} \text{ s } e^- \text{ at } r = 10 R_E$

Bounce

$$\text{bounce time} \equiv T_{\text{bounce}} = \oint \frac{ds}{v_{||}}$$

$$ds = (dr^2 + r^2 d\lambda^2)^{1/2} = r_e \cos \lambda (1 + 3 \sin^2 \lambda)^{1/2} d\lambda$$

$$v_{||} = (v^2 - v_{\perp}^2)^{1/2}$$

$$\text{but } \frac{v_{\perp}(r, \lambda)}{B(r, \lambda)} = \frac{v^2 \sin^2 \alpha_e}{B_e} = \text{const}$$

where α_e and B_e are equatorial pitch angle and field strength

$$\text{so } v_{||} = v \left(1 - \frac{\sin^2 \alpha_e B(r, \lambda)}{B_e} \right)^{1/2}$$

$$B(r, \lambda) = \frac{m}{r^3} (1 + 3 \sin^2 \lambda)^{1/2} \quad (\text{Dipole field})$$

$$= \frac{B_e}{\cos^6 \lambda} (1 + 3 \sin^2 \lambda)^{1/2}$$

(because
 $r = r_e \cos^2 \lambda$
field line equation)

numerically integrating

$$T_{\text{bounce}} \sim 4 \frac{r_e}{v} T(\alpha_e)$$

where $T(\alpha_e) = 1.3 - 1.56 \sin \alpha_e$
(not a strong dependence on α_e)

for $v = c$ and $T(\alpha) = 1$

$$r_e \quad 1.5 r_e \quad 3 r_e \quad 6 r_e$$

$$T_{\text{bounce}} \quad .13 s \quad .26 s \quad .52 s$$

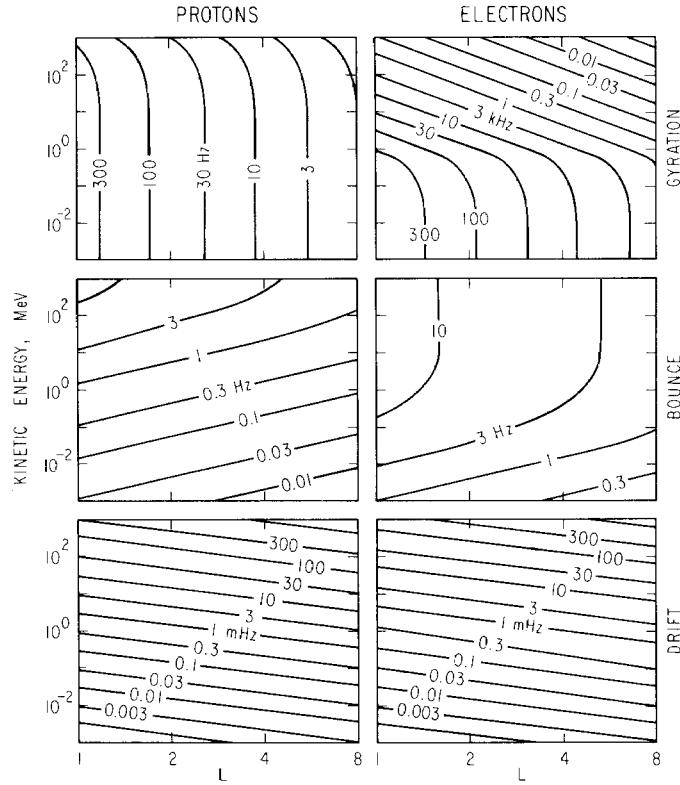


Fig. 6. Contours of constant adiabatic gyration, bounce, and drift frequency for equatorially mirroring particles in a dipole field. Adiabatic approximation fails in upper right-hand corners ($E \sim 1 \text{ GeV}$, $L \sim 8$), since $\Omega_1 \sim \Omega_2 \sim \Omega_3$ implies $|\varepsilon| \sim 1$.