Collective Description of the Plasma

- **Remind about Lecture 4 definition of distribution function**

Liouville's Theorem

Force acting on a particle = $\vec{F} = m\vec{g} + Ze(\vec{E} + \vec{v}\times\vec{B}) = m d\vec{v}/dt$

Assume fields change sufficiently smoothly so that $\vec{E}(\vec{r} + d\vec{r}, t) - \vec{E}(\vec{r}, t) \approx$ **infinitessimal.**

Then conservation of particles implies that $f(\vec{r}, \vec{v}, t)$ **obeys a continuity equation:**

$$\frac{\partial f}{\partial t} + \vec{\nabla}\cdot\vec{v}f + \vec{\nabla}_v\cdot\frac{\vec{F}}{m}f = 0$$

where $\vec{\nabla} \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$ **and** $\vec{\nabla}_v = \hat{x}\frac{\partial}{\partial v_x} + \hat{y}\frac{\partial}{\partial v_y} + \hat{z}\frac{\partial}{\partial v_z}$

Therefore: $\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla}f + \frac{\vec{F}}{m}\cdot\vec{\nabla}_v f + f\left[\vec{\nabla}\cdot\vec{v} + \vec{\nabla}_v\cdot\frac{\vec{F}}{m}\right] = 0$

but $\vec{\nabla}\cdot\vec{v} = 0$ **(because** $\vec{v}$ and $\vec{r}$ **are independent variables)**

$\vec{\nabla}_v\,(m\vec{g} + Ze\vec{E}) = 0$ **because** $\vec{g}$, $\vec{E}$ **do not depend on** $\vec{v}$**, and**

$\vec{\nabla}_v\cdot(\vec{v}\times\vec{B}) = \vec{B}\cdot\vec{\nabla}_v\times\vec{v} - \vec{v}\cdot\vec{\nabla}_v\times\vec{B} = 0$ **where we note that** v_x, v_y, and v_z **are independent variables.**

So, $$\frac{df}{dt} = \frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla}f + \frac{\vec{F}}{m}\cdot\vec{\nabla}_v f = 0$$

Liouville's Theorem

Now, back to handwritten ...

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial t} + \bar{v}\cdot\nabla f + \frac{\bar{F}}{m}\cdot\nabla_v f = 0}$$

Liouville's Theorem or

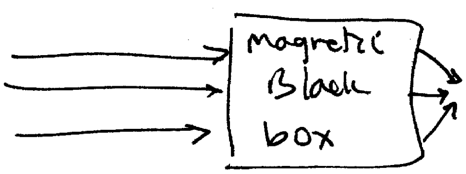
~~Collisionless Boltzman Equation~~

Another formulation of Liouville's theorem:
Let $d^3r\, d^3v$ be volume in $\vec{r}-\bar{v}$ space (i.e. phase space) occupied by a given set of particles.

Then $f\, d^3r\, d^3v$ = number of particles = constant
but f = const. along trajectory
$\therefore$ $d^3r\, d^3v$ = constant following trajectory

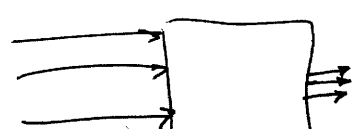
$d^3r\, d^3v = d^3r \frac{v}{m}\, d\mathcal{E}\, d\Omega$ and $d^3r = dA\, v\, dt$

<u>Example:</u>
Magnetic Focusing no electric Fields $\therefore$ v = const
$\therefore$ $dA\, d\Omega$ = const



dA reduced, $\therefore$ $d\Omega$ increased

a device which does this



is <u>impossible</u>
no matter how complex the fields

Example: increase energy of all particles $\Rightarrow$ $\Delta\mathcal{E}$ increases, $\therefore$ d^3r must decrease $\rightarrow$ i.e. plasma must be <u>compressed</u>

now Liouville $\rightarrow$ Vlasov

If there were collisions (which cause discontinuous changes in $\bar{v}$ and therefore make a particle "jump" from one area in phase space to another, or $\bar{E}, \bar{B}$ fields which vary appreciably within $d\bar{r}$, then continuity equation becomes

$$\frac{\partial f}{\partial t} + \nabla \cdot \vec{v} f + \nabla_v \cdot \left(\frac{\bar{F}}{m} f \right) = \left(\frac{\delta f}{\delta t} \right)_c$$ Boltzman equation

$\vec{F}$ includes only the effect of smooth fields (as discussed above)

The form of $\left.\frac{\delta f}{\delta t}\right)_c$ must be determined from a theory of the collision or other process involved.

$\left(\frac{\delta f}{\partial t} \right)_c$ may include:

(1) binary collisions
(2) microscopic $\bar{E}$ and $\bar{B}$ fluctuations
(3) radiation by single particles
(4) creation and annihilation
(5) ionization, recombination and charge exchange

Neglecting these effects then f is determined by the collisionless Boltzman Equation (Vlasov Equation)

$$\boxed{\frac{\partial f}{\partial t} + \bar{v} \cdot \bar{\nabla} f + \frac{Ze}{m} \left(\bar{E} + \bar{v} \times \bar{B} \right) \cdot \bar{\nabla}_v f = 0}$$

Now, E and B are determined by Maxwells equations

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho_c}{\epsilon_0} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

(treat all charges and currents explicitly — do NOT introduce $\vec{D}$ and $\vec{H}$)

$$\rho_c = \text{charge density} \equiv \sum_a Z_a e \int d^3v\, f_a$$

$$\vec{J} = \sum_a Z_a e \int d^3v\, \vec{v} f_a = \text{Current Density}$$

Sum over particle species

$Z_a = -1$ for electrons

$= +1$ " protons etc

These equations, plus Vlasov equation are the complete set of equations describing a collisionless plasma.

Collective Description of Plasma cont'd

$f_a(\bar{r},\bar{v},t)$ distribution function of species a

m_a mass of particle of species a

$Z_a e$ charge " " " "

($z = -1$ for electrons)

then

$$n_a = \int d^3v\, f_a \quad \text{number density of species a}$$

$$n_a \vec{V}_a = \int d^3v\, \bar{v}\, f_a \quad \underline{\text{number flux density}}$$

where $\bar{V}_a$ bulk velocity, or center-of-mass velocity of species a

> Mass density of plasma as a whole:

$$\rho = \sum_a m_a n_a = \sum_a m_a \int d^3v\, f_a$$

(note we will use ρ_c for charge den

> momentum density of plasma: (mass flux density)

$$\rho \vec{V} = \sum_a m_a n_a \vec{V}_a = \sum_a m_a \int d^3v\, \bar{v}\, f_a$$

$\bar{V}$ = bulk velocity or center-of-mass velocity of entire plasma (in general not coincident with any of the $\bar{V}_a$)

> Electric current density

$$\bar{J} = \sum_a Z_a e\, n_a \bar{V}_a = \sum_a Z_a e \int d^3v\, \bar{v}\, f_a$$

Kinetic Tensor $\overleftrightarrow{K}_a = \int d^3v\, f_a\, \bar{p}\,\bar{v} = \int d^3v\, f_a m_a \bar{v}\,\bar{v}$

$\bar{p}\,\bar{v}$ is the direct product (NOT the dot product) of the two vectors

$\overleftrightarrow{K}$ represents set of 9 quantities

$$\begin{pmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{pmatrix}$$

notation: $K_{\text{row column}}$

where $K_{xx} = \int d^3v\, f\, p_x v_x$, $K_{xy} = \int d^3v\, f\, p_x v_y$ etc...

$\overleftrightarrow{K}$ can also be written $K_{ij} = \int d^3v\, f\, p_i v_j$ $\quad i = 1,2,3$, $j = 1,2,3$ (or also $i = x,y,z$ $\quad j = x,y,z$)

Note only 6 of the 9 components are independent since $K_{ij} = K_{ji}$ ($\overleftrightarrow{K}$ is a <u>symmetric</u> tensor)

unit tensor $\overleftrightarrow{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \delta_{ij}$

where $\delta_{ij} = 1$ for $i = j$ (Kronecker delta)

Physical significance of kinetic tensor:

Let $\hat{n}$ = normal to some surface

Then the vector $\overleftrightarrow{K}_a \cdot \hat{n} = \int d^3v\, f_a\, \vec{p}\,(\bar{v}\cdot\hat{n})$

continued →

$\overleftrightarrow{K}_a \cdot \hat{n}$ then represents the <u>flux density of momentum crossing the surface</u>

Thus, in particular with $\hat{n} = \hat{x}$

K_{xx} = flux density of x component of momentum through a surface normal to the $\hat{x}$ axis

K_{yx} = flux density of y component of momentum through a surface normal to the $\hat{x}$ axis

etc ...

Note that if $\overleftrightarrow{T}$ is some tensor (not necessarily symmetric) and $\bar{u}$ some vector, then

$$\overleftrightarrow{T} \cdot \bar{u} = (T_{xx}u_x + T_{xy}u_y + T_{xz}u_z)\hat{x} + (T_{yx}u_x + T_{yy}u_y + T_{yz}u_z)\hat{y} + (T_{zx}u_x + T_{zy}u_y + T_{zz}u_z)\hat{z}$$

$$\bar{u} \cdot \overleftrightarrow{T} = (T_{xx}u_x + T_{yx}u_y + T_{zx}u_z)\hat{x} + (T_{xy}u_x + T_{yy}u_y + T_{zy}u_z)\hat{y} + (T_{xz}u_x + T_{yz}u_y + T_{zz}u_z)\hat{z}$$

$$\bar{u} \cdot \overleftrightarrow{T} \neq \overleftrightarrow{T} \cdot \bar{u} \quad \text{if } T \text{ is not symmetric}$$

Shorter notation $\overleftrightarrow{T} \cdot \bar{u} \rightarrow T_{ij} u_j$ (repeated indizies summed over.) (Einstein convention)

$\vec{u} \cdot \overleftrightarrow{T} \rightarrow T_{ji} u_j$

if $\bar{v}$ is another vector then the scalar

$\bar{v} \cdot \overleftrightarrow{T} \cdot \bar{u} = v_i T_{ij} u_j$ (summed over i and j understood) also $\overline{u} \cdot T \cdot \overline{v} = u_i T_{ij} v_j$

in general $\overline{v} \cdot \overleftrightarrow{T} \cdot \overline{u} \neq \overline{u} \cdot T \cdot \overline{v}$

Kinetic tensor is defined in some frame of reference.

The particular $\overleftrightarrow{K}_a$ defined in the frame of reference moving with the center-of-mass velocity of the <u>whole</u> <u>plasma</u> is called

The <u>PRESSURE Tensor</u>

$$\overleftrightarrow{P}_a = \int d^3v\, f_a\, \overline{p}^*\, \overline{v}^* \quad \text{where}$$

$$\overline{p}^* = \overline{p} - m_a \overline{V}$$
$$\overline{v}^* = \overline{v} - \overline{V}$$

($\overline{V}$ defined on page 1)

(from now on $*$ will designate quantities referenced to the plasma center-of-mass frame of refence)

$\overleftrightarrow{P}_a$. partial pressure of species a

$$\overleftrightarrow{P} = \sum_a \overleftrightarrow{P}_a = \sum_a m_a \int d^3v\, f_a\, \overline{v}^*\, \overline{v}^*$$

pressure tensor of whole plasma

Note that partial pressure $\overleftarrow{P}_a$ is defined with respect to the center-of-mass frame of the entire plasma, not the center-of-mass frame of species a.

It is possible to define a pressure tensor for species a with respect to the center of mass frame of species a; but then the different $\overleftarrow{P}_a$ are defined WRT different frames and cannot be added to get pressure tensor of entire plasma.

EXAMPLES of special pressure tensors

(A) If $f(\bar{r},\bar{v},t)$ is isotropic (i.e. $f(\bar{r},\bar{v},t) = f(\bar{r},v,t)$)

then $m\int d^3v\, f v_x^2 = m\int d^3v\, f v_y^2 = m\int d^3v\, f v_z^2 \equiv P$

$\int d^3v\, f\, v_x v_y = 0 \qquad \int d^3v\, f\, v_x v_z = 0$ etc

then $\overleftarrow{P} = \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{pmatrix} = P\,\overleftarrow{1}$

where P is a scalar or isotropic pressure (corresponding to the ordinary notion of pressu

(B) If f_a has axial symmetry about (say) the magnetic field direction $\hat{B}$ and if $\hat{B} = \hat{z}$

$$m\int d^3v\, f v_x^2 = m\int d^3v\, f v_y^2 \equiv P_\perp$$

$$m\int d^3v\, f v_z^2 \equiv P_\parallel$$ in general $P_\parallel \neq P_\perp$

$$\int d^3v\, f v_x v_y = 0 \quad \text{etc}$$

then $\overleftarrow{P} = \begin{pmatrix} P_\perp & 0 & 0 \\ 0 & P_\perp & 0 \\ 0 & 0 & P_\parallel \end{pmatrix}$ to write this in any coordinant system ($\vec{B}$ not necessarily along $\hat{z}$) rewrite

$$\overleftrightarrow{P} = \begin{pmatrix} P_\perp & 0 & 0 \\ 0 & P_\perp & 0 \\ 0 & 0 & P_\perp \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & P_\parallel - P_\perp \end{pmatrix}$$

$$= P_\perp \overleftarrow{1} + (P_\parallel - P_\perp)\frac{\vec{B}\vec{B}}{B^2}$$

or $P_{ij} = P_\perp \delta_{ij} + (P_\parallel - P_\perp)\dfrac{B_i B_j}{B^2}$ in any coordinant system

note $p_\perp v_\perp = p_x v_x + p_y v_y$ if $\hat{B} = \hat{z}$

but $P_\perp = \int d^3v\, f\, p_x v_x = \int d^3v\, f\, p_y v_y$

$\therefore\ P_\perp = \int d^3v\, f\, \frac{1}{2} p_\perp v_\perp$ $\Big\}$ $P_\perp$ = perpendicular pressure

$P_\parallel$ = parallel pressure

(c) for a Maxwellian distribution

$$\overleftrightarrow{P} = nkT \overleftrightarrow{1}$$

Next: Summation of single particle guiding center motions to obtain electrical current.