

# Lecture 18

Review MHD Equations

8 eqns, 8 unknowns

Today: (1) Comparison of  $\mathbf{v} \times \mathbf{B}$  and  $\frac{\partial \mathbf{H}}{\partial t}$  terms in Ohm's law

$\Rightarrow$  Magnetic Reynolds #

(2) Discuss meaning of motion of B-Field Lines

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Comparison of Finite Resistivity term  
in Ohm's law to  $\mathbf{v} \times \mathbf{B}$  term:

assume  $\sigma = \text{constant}$ , then take

curl of Ohm's law  $\nabla \times (\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \frac{\mathbf{J}}{\sigma}$

and use Faraday's Law and the

vector identity  $\nabla \times (\mathbf{v} \times \mathbf{B}) = -\nabla^2 \mathbf{B} + \nabla(\nabla \cdot \mathbf{B})$

to obtain

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B} \quad (\text{Parks 5.58})$$

Order of Magnitude comparison of  
1<sup>st</sup> to 2<sup>nd</sup> term

$$\frac{VB}{L} \left( \frac{B}{\mu_0 L^2} \right)^{-1} = L V \mu_0 \sigma \approx R_m$$

Magnetic Reynolds #

If  $R_m \ll 1$  Then

$$\frac{\partial \vec{B}}{\partial t} = \frac{\nabla^2 \vec{B}}{\mu_0 \sigma}$$

Diffusion of  
magnetic field

If  $R_m \gg 1$  then

$$\frac{\partial \vec{B}}{\partial t} = \vec{v} \times (\vec{v} \times \vec{B})$$

Transport of  
magnetic field  
by flow

also for  $R_m \gg 1$

Ohm's law can be written

$$\vec{E} + \vec{v} \times \vec{B} = 0$$

If  $R_m \gg 1$  plasma behaves as if  
the conductivity were  $\infty$ .

$R_m$  can be large either because  
 $\sigma$  is large or because  $L$  is large (as in  
astrophysics or space applications)

for comparison, a fully ionized hydrogen  
plasma in thermal equilibrium has a temperature  $T$   
and  $\sigma = 10^{17} \text{ sec}^{-1} \left( \frac{kT}{1 \text{ keV}} \right)^{3/2}$

for Copper, by comparison

$$\sigma \sim 10^{18} \text{ sec}^{-1}$$

Consequences of  $R_m$  large and  $\bar{\mathbf{E}} + \bar{\mathbf{v}} \times \bar{\mathbf{B}} = 0$

①  $\bar{\mathbf{E}} \cdot \bar{\mathbf{B}} = 0$        $\bar{\mathbf{E}} \cdot \bar{\mathbf{v}} = 0$   
 in steady state  $\bar{\mathbf{E}} = -\nabla\Phi$

$\therefore \bar{\mathbf{B}} \cdot \nabla\Phi = 0$        $\bar{\mathbf{v}} \cdot \nabla\Phi = 0$

$\therefore$ , magnetic field lines and plasma flow lines are EQUIPOTENTIALS

②  $\bar{\mathbf{v}}_{\perp} = \frac{\bar{\mathbf{E}} \times \bar{\mathbf{B}}}{B^2}$        $\therefore \bar{\mathbf{v}} = \bar{\mathbf{v}}_{\perp} + v_{||} \frac{\bar{\mathbf{B}}}{B}$

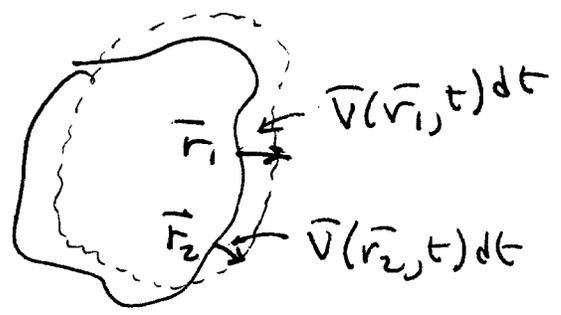
$v_{||}$  is NOT determined by this equation

③ Magnetic flux through any loop moving with the plasma remains constant

④ Two points moving with ~~the plasma~~ <sup>the plasma</sup> ~~on~~ a field ~~line~~ <sup>line</sup> remain on a field line

Proof ③ & ④ over

③ Prove that magnetic flux through any loop moving with the plasma remains constant



$$\Phi_m = \int d\vec{S} \cdot \vec{B}$$

$$\begin{aligned} \frac{d}{dt} \Phi_m &= \frac{d}{dt} \int d\vec{S} \cdot \vec{B} = \int d\vec{S} \cdot \frac{\partial \vec{B}}{\partial t} + \oint \underbrace{\vec{v} \times d\vec{l}}_{\substack{\text{area swept out} \\ \text{by motion of loop}}} \cdot \vec{B} = \\ &= \int d\vec{S} \cdot \frac{\partial \vec{B}}{\partial t} - \oint d\vec{l} \cdot (\vec{v} \times \vec{B}) = \int d\vec{S} \cdot \left[ \frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{v} \times \vec{B}) \right] = \\ &= - \oint d\vec{l} \cdot [\vec{E} + \vec{v} \times \vec{B}] = 0 \end{aligned}$$

④ Prove that two points moving with the plasma on a field line remain on a field line

$$\begin{aligned} \vec{s} &= \vec{r}_2 - \vec{r}_1 & \frac{d\vec{s}}{dt} &= \vec{v}_2 - \vec{v}_1 \rightarrow (\vec{s} \cdot \nabla) \vec{v} \\ & & & \approx |\vec{s}| \rightarrow 0 \\ \frac{d}{dt} \vec{s} \times \vec{B} &= \frac{d\vec{s}}{dt} \times \vec{B} + \vec{s} \times \frac{d\vec{B}}{dt} \\ &= -\vec{B} \times (\vec{s} \cdot \nabla) \vec{v} + \vec{s} \times \left( \frac{\partial \vec{B}}{\partial t} + (\vec{v} \cdot \nabla) \vec{B} \right) \end{aligned}$$

but  $\rightarrow$

proof (4) contd

But if  $\vec{E} = -\vec{v} \times \vec{B}$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) = -(\vec{v} \cdot \nabla) \vec{B} + (\vec{B} \cdot \nabla) \vec{v} - \vec{B} (\nabla \cdot \vec{v})$$

so  $\frac{d}{dt} \vec{\delta} \times \vec{B} = -(\vec{\delta} \times \vec{B}) \nabla \cdot \vec{v} + \vec{\delta} \times (\vec{B} \cdot \nabla) \vec{v} - \vec{B} \times (\vec{\delta} \cdot \nabla) \vec{v}$

write  $\vec{\delta} = \delta_{\parallel} \hat{B} + \vec{\delta}_{\perp} = \delta_{\parallel} \hat{B} + \vec{B} \times (\vec{\delta} \times \hat{B})$

$$\begin{aligned} \therefore \frac{d}{dt} \vec{\delta} \times \vec{B} &= -(\vec{\delta} \times \vec{B}) \nabla \cdot \vec{v} + \vec{\delta}_{\perp} \times (\vec{B} \cdot \nabla) \vec{v} - \vec{B} \times (\vec{\delta}_{\perp} \cdot \nabla) \vec{v} \\ &= -(\vec{\delta} \times \vec{B}) \nabla \cdot \vec{v} + [\vec{B} \times (\vec{\delta} \times \vec{B})] \times (\vec{B} \cdot \nabla) \vec{v} - \vec{B} \times \left[ \vec{B} \times (\vec{\delta} \times \vec{B}) \cdot \nabla \right] \vec{v} \end{aligned}$$

This is of the form  $\frac{d}{dt} \vec{\delta} \times \vec{B} = \vec{\delta} \times \vec{B} \cdot \vec{T}$

where  $\vec{T}$  is a tensor independent of  $\vec{\delta}$

$$\text{thus } \frac{d^2}{dt^2} \vec{\delta} \times \vec{B} = \left[ \frac{d}{dt} (\vec{\delta} \times \vec{B}) \right] \cdot \vec{T} + (\vec{\delta} \times \vec{B}) \cdot \frac{d}{dt} \vec{T} =$$

$$= (\vec{\delta} \times \vec{B}) \cdot \left[ \vec{T} \vec{T} + \frac{d}{dt} \vec{T} \right]$$

and can similarly show all higher order time derivatives of  $\vec{\delta} \times \vec{B}$  are proportional to  $\vec{\delta} \times \vec{B}$

∴ if  $\vec{\delta} \times \vec{B} = 0$  at  $t=0$  (ie  $\vec{\delta}$  parallel to  $\vec{B}$ )

then  $\frac{d}{dt} \vec{\delta} \times \vec{B} = 0$ ,  $\frac{d^2}{dt^2} \vec{\delta} \times \vec{B} = 0$ ,  $\frac{d^3}{dt^3} \vec{\delta} \times \vec{B} = 0$  etc at  $t=0$

∴  $\vec{\delta} \times \vec{B} = 0$  at all  $t \rightarrow$  ie  $\vec{\delta}$  remains parallel to  $\vec{B}$ . QED

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thus, if we consider a set of plasma elements on a field line at time  $t=0$  as tracing out that field line, at all other times these elements will be on a field line and can be thought of as tracing out a field line

∴ Can think of plasma motion as being also the motion of field lines

# Discussion of Meaning of moving B lines

Confusing topic not studied in standard plasma physics texts.

Do Field Lines Really Move?

meaningless question - depends on point of view

Maxwell never thought about field lines

(we can write down Maxwell's eqns, force laws...

with  $\vec{B} = \vec{F}(x, y, z, t)$  and solve all

problems in E+M without talking about field lines.

They are introduced as crutches to help you think about the problem.

Properties of Field Lines

1. point in direction of  $\vec{B}$  or  $\vec{E}$

2. density of lines  $\propto$  field strength

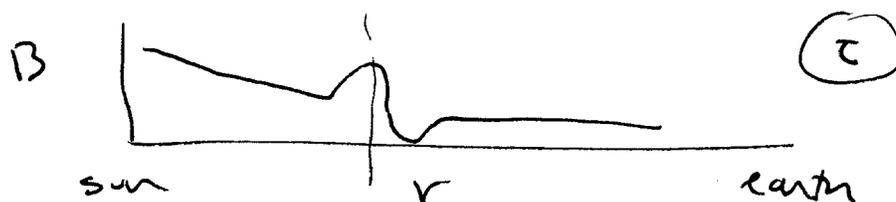
can add 3<sup>rd</sup> property That lines move with  $\vec{v} = \frac{\vec{E} \times \vec{B}}{B^2}$  if this

doesn't violate properties above. It Doesn't as we will see

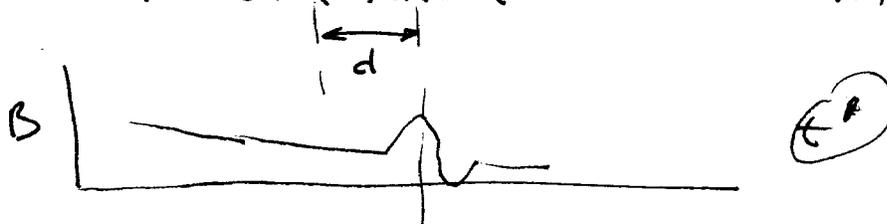
Is it useful in understanding these problems to think of moving field lines or not

## Example with Solar wind B

suppose we have at some time  $t$



know that at some later time  $t'$  we will have



$$\text{where } d = \frac{E}{B} (t' - t)$$

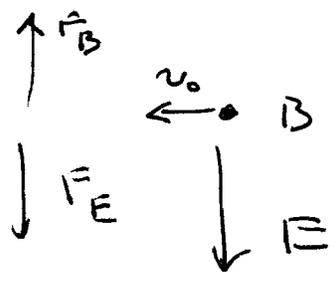
Easy to say this result in terms of moving field lines. We could get the same  $B(r, t')$  from  $B(r, t)$  from Maxwell's equations without talking about field lines not in. But This Way is Easier.

## Another example

Single particle in constant  $E + B$  field

$$\text{with } \vec{v} = \frac{\vec{E} \times \vec{B}}{B^2} \text{ at } t=0$$

we have



$$\vec{F}_B = e v B$$

$$= e \frac{E}{B} B = e E = -F_E$$

∴ No Net Force → particle moves to left at constant velocity

We could talk about this using field lines in 2 ways: (lines are straight + uniform in both cases)

Case 1: E + B lines fixed and particle moves across both lines

Case 2: particle gyrates about  $\vec{B}$  lines in all situations. In above case  $\vec{B}$  lines and particle move to left.

preferred

How to Measure motion of B-lines →

Why is Case 2 preferred?

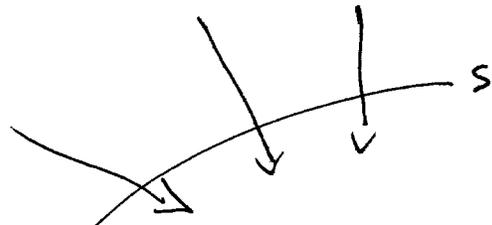
- I. Same terminology works well in all limits (ie. with or without  $\vec{E}$ ) That is, without  $\vec{E}$  particle still gyrates around  $\vec{B}$
- II. Easiest way to understand the problem ~~when~~ such as solar wind case

## Rigorous Proof of Field Line motion

Q: Prove that Field line motion with

$$\vec{V}_B = \frac{\vec{E} \times \vec{B}}{B^2} \text{ keeps magnitude of } B \text{ correct}$$

take surface  $S$  having  $\vec{B} \perp$  to surface



must show conservation equation

$$\text{ie } \frac{dB}{dt} + \nabla_s \cdot B \vec{V}_B = 0$$

ie, show field lines flow at just the right rate that  $B$  is changing so density of lines  $\propto$  field strength because of flow.

Set up coordinates with  $\vec{B}$  in  $\hat{z}$  direction  
at some point  $\vec{B} \cdot \vec{V}_B \equiv B \frac{\vec{E} \times \vec{B}}{B^2}$  surface  $S$  in  $x-y$  plane

$$B \vec{V}_B)_x = E_y$$

$$B \vec{V}_B)_y = -E_x$$

$$\therefore \nabla_s \cdot \vec{V}_B B = \frac{\partial}{\partial x} E_y - \frac{\partial}{\partial y} E_x = \nabla \times E)_z = -\frac{\partial t}{\partial z}$$

since  $\vec{B} = B \hat{z}$

so conservation equation is obeyed.

Note: we can select  $\vec{V}_B = \frac{\vec{E} \times \vec{B}}{B^2} + \vec{v}'$

for any  $\vec{v}'$  having  $\nabla \cdot \vec{B} \vec{v}' = 0$   
so there are motions besides

$\vec{V}_B = \frac{\vec{E} \times \vec{B}}{B^2}$  that keep proper field strength.

We already showed (pages 4 & 5 above) that field lines moving with the plasma where  $\vec{E} = -\vec{v} \times \vec{B}$  ( $\sigma \rightarrow \infty$ ) limit (equivalent to  $\vec{V} = \frac{\vec{E} \times \vec{B}}{B^2}$ )

~~have~~ maintain the right direction of the field.

So: motion of field lines with

$\vec{V}_B = \frac{\vec{E} \times \vec{B}}{B^2}$  maintains field strength & direction

