

1. Finish discussion of ionospheric currents (See Last Notes)  
understand  $\sigma_p, \sigma_{ff}, \sigma_o$  in  $\overleftarrow{\sigma_2}$   
 Examples from ionosphere and data

2. Start plasma wave discussion  
 MHD Waves  
 $\rightarrow$  Alfvén waves

#### MHD Assumptions

1. in Generalized Ohm Law, inertial and pressure gradient terms are negligible and  $\sigma \rightarrow \infty \therefore \vec{E} + \vec{V} \times \vec{B} = 0$
2. Pressure tensor has no off-diagonal terms and the diagonal terms are equal  
 $\therefore$  No viscosity and pressure isotropic
3. Dissipative terms in energy equation are negligible ( $E^* \cdot J^* = 0$ , heat flux = 0)  
 $\therefore$  no ohmic heating & no heat flux
4. Charge neutrality  
 $\therefore p_c = \epsilon \rho = \nabla \cdot E = \frac{\partial P}{\partial t} = 0$

Review  $\Rightarrow$  MHD equations

Now what about a plasma?

Review

### MHD EQUATIONS

$$\frac{d}{dt} \rho + \rho \bar{\nabla} \cdot \bar{v} = 0$$

Verdict (

$$\rho \frac{d\bar{v}}{dt} + \bar{\nabla} P = -\frac{1}{\mu_0} \bar{B} \times (\bar{\nabla} \times \bar{B})$$
$$\frac{d\bar{B}}{dt} = \bar{\nabla} \times (\bar{v} \times \bar{B}) ; \quad \bar{\nabla} \cdot \bar{B} = 0$$

$$\frac{d}{dt} \frac{P}{\rho} = 0$$

where  $\frac{d}{dt} \equiv \frac{D}{Dt} + \bar{v} \cdot \bar{\nabla}$

WAVE Motion: Assume small departures  
from uniform state

$$\rho = \rho_0 + \delta\rho$$

$$\bar{v} = \frac{D}{Dt} \bar{v}$$

$$\bar{B} = \bar{B}_0 + \delta\bar{B}$$

$$P = P_0 + \delta P$$

$\rho_0, \bar{B}_0, P_0$  constant

in space & time

$\delta\rho, \delta\bar{v}, \delta\bar{B}, \delta P$

infinitesimally small

linearize MHD equations

$$\text{Let } \frac{\vec{x}}{u} = \frac{\vec{k}}{\omega} \quad \text{and } B_z = 0$$

i.e. choose  $\vec{x}$  as direction of propagation of wave and the undisturbed  $B_0$  is in  $x-y$  plane

where  $u$  = phase speed of wave

$$\vec{u} = u \frac{\vec{k}}{\omega} = \text{phase velocity}$$

Then: MHD linearized can be shown to be:

$$-u \delta p + \rho \delta v_x = 0 \quad (1)$$

$$-u \rho \delta v_x + \delta P + \frac{B_y \delta B_y}{4\pi} = 0 \quad (2)$$

$$-u \rho \delta v_y - \frac{B_x \delta B_y}{4\pi} = 0 \quad (3)$$

$$-u \rho \delta v_z - \frac{B_x \delta B_z}{4\pi} = 0 \quad (4)$$

$$-u \delta B_y - B_x \delta v_y + B_y \delta v_x = 0 \quad (5)$$

$$-u \delta B_z - B_x \delta v_z = 0 \quad (6)$$

$$u (\delta P - \gamma \frac{P}{\rho} \delta \rho) = 0 \quad (7)$$

$$\delta B_x = 0 \quad (8)$$

$\Rightarrow 8 \text{ eqns}, 8 \text{ unknowns} \therefore \text{Determinant of coefficients} = 0$

Note that  $u$  is independent of  $\omega$   
 $u$  don't appear surface with amplitude  
 $\Rightarrow$  MHD waves are non-dispersive

$\Rightarrow$  All ~~different~~ frequencies propagate at same speed

$\therefore$  Consider only sinusoidal waves

then superimpose to get arbitrary wave shape

examples

Case A : Assume  $u=0$ ,  $B_x=0$

then from (1)  $\delta v_x = 0$

(3), (4), (5), (6), (7) satisfied identically

$$\text{from (2)} \quad \delta P + \frac{B_y \delta B_y}{4\pi} = \delta \left( P + \frac{B_y^2}{8\pi} \right) = 0$$

$\therefore \sin \delta v_x = \delta B_x = 0$  only for uniform  $B$

$$\delta \left( P + \frac{B_y^2}{8\pi} \right) = 0 \quad (\text{other arbitrary})$$

Entropy discontinuity (type 2)

$\Rightarrow$  important for Magnetopause

(osिवल वेव्स - not finite amplitude)

case B  $u \neq 0$  look at  $\vec{z}$ -components

(4) and (6)

$$u \rho \delta V_z + \frac{B_x}{4\pi} \delta B_z = 0$$

$$B_x \delta V_z + u \delta B_z = 0$$

non trivial solution  
only if

$$\begin{vmatrix} u & \frac{B_x}{4\pi} \\ B_x & u \end{vmatrix} = 0$$

$$\alpha' u^2 = \frac{B_x^2}{4\pi \rho} = b_x^2$$

$$\text{where } b^2 = \frac{B^2}{4\pi \rho} \\ (b = \text{Alfven speed})$$

$$\text{in general } u^2 = b_x^2$$

will not make other  $6 \times 6$  determinant = 0

i.e. all other  $\delta$ 's = 0

$$\therefore \begin{cases} \delta V_y = \delta V_z = 0 & \delta B_y = 0 & \delta \rho = C = SP \\ u = \pm \sqrt{\frac{B_x^2}{4\pi \rho}} & \delta V_z = -u \frac{\delta B_z}{B_x} = \mp \sqrt{\frac{\delta B_z}{4\pi \rho}} \end{cases}$$

called Alfvén wave or Intermediate wave

Case C Assume  $u^2 \neq b_x^2$

then from (4) and (6)  $\delta B_z = 0$   $\delta V_z = 0$

Define  $\alpha^2 = \gamma \frac{P}{\rho}$  Then  $SP = \alpha^2 \delta \rho$

from (7)

from (1)  $\delta V_x = u \frac{SP}{\rho}$  insert into (2)

$$SP(\alpha^2 - u^2) + \frac{B_x \delta B_y}{4\pi} = 0$$

eliminate  $\delta V_y$  between (3) and (5) to obtain  $\rightarrow$

$$\frac{\delta P}{P} u^2 B_y + \delta B_y \left( \frac{B_x^2}{4\pi P} - u^2 \right) = 0$$

Thus, 2 equations for  $B_y \frac{\delta P}{P}$  and  $\delta B_y$ :

$$B_y \frac{\delta P}{P} u^2 - \delta B_y (u^2 - b_x^2) = 0 \quad \left. \begin{array}{l} \\ \end{array} \right\}$$

$$B_y \frac{\delta P}{P} (u^2 - a^2) - \delta B_y (b_y^2) = 0 \quad \left. \begin{array}{l} \\ \end{array} \right\}$$

$$\frac{B_y^2}{4\pi P} = b_y^2 \quad \therefore \quad \begin{vmatrix} u^2 & -(u^2 - b_x^2) \\ u^2 - a^2 & -b_y^2 \end{vmatrix} = 0$$

$$\text{or } \boxed{(u^2 - a^2)(u^2 - b_x^2) = u^2 b_y^2} \quad \text{eqn (9)}$$

$$b_z = 0$$

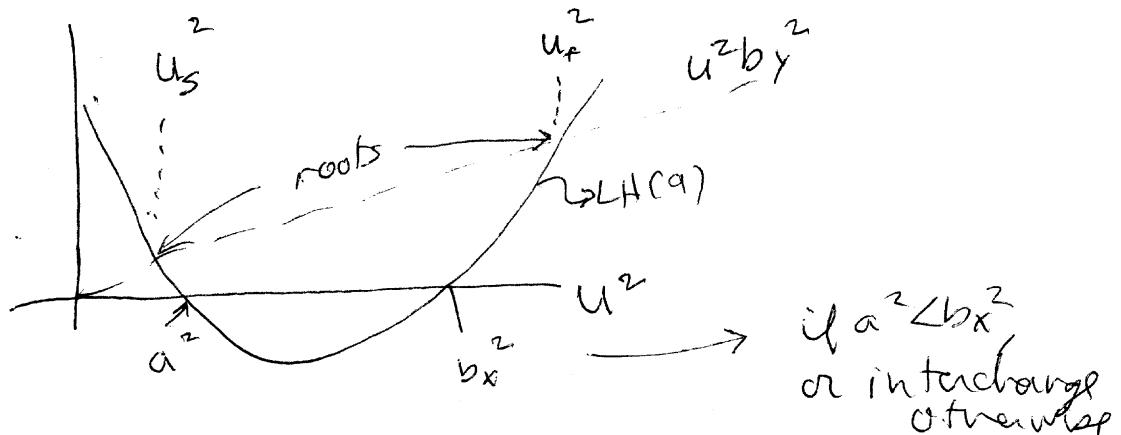
$$\text{multiplying out } u^4 - u^2(a^2 + b_x^2 + b_y^2) + a^2 b_x^2 = 0$$

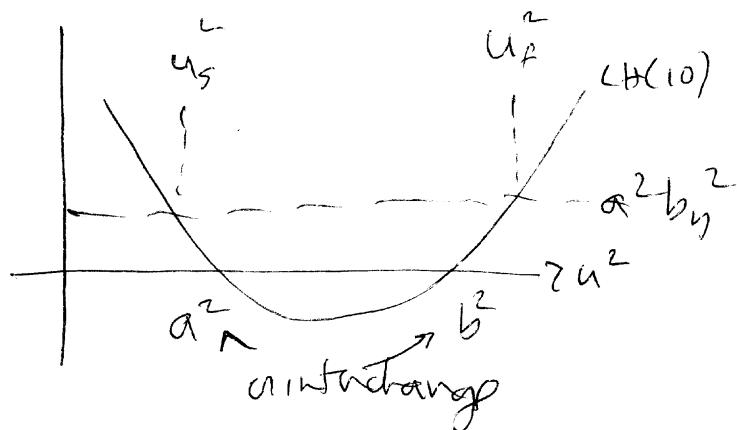
$$\text{or } u^4 - u^2(a^2 + b^2) + a^2 b^2 - a^2 b_y^2 = 0$$

$$\text{or } \boxed{(u^2 - a^2)(u^2 - b^2) = a^2 b_y^2} \quad \text{eqn (10)}$$

(10) equivalent to (9) : quadratic equation for  $u^2$   
2 roots fast and slow waves

consider LHS sides (Left hand sides) of (9)+(10)





i. for 1 root ( $u_s$ ) both factors on L.H side negative

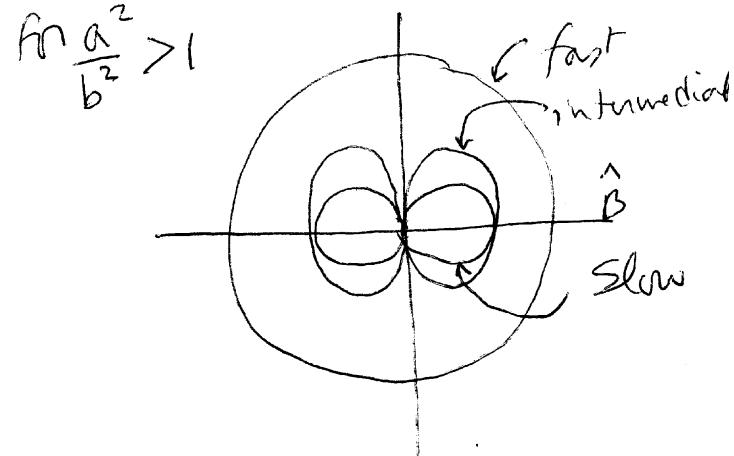
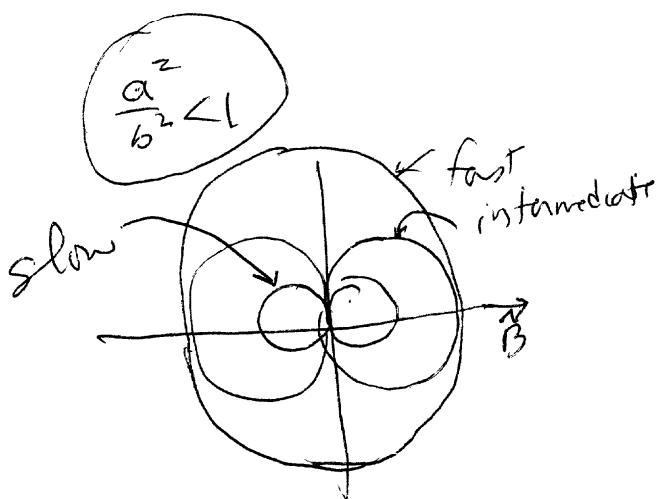
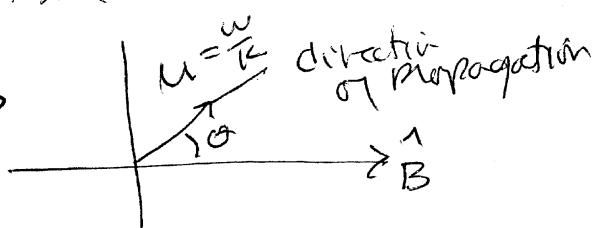
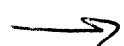
for other root ( $u_f$ ), both positive

$$\begin{aligned} i. \quad u_s^2 &\leq a^2 \\ u_f^2 &\geq a^2 \end{aligned}$$

$$\begin{aligned} u_s^2 &\leq b_x^2 \leq b^2 \\ u_f^2 &\geq b^2 \geq b_x^2 \end{aligned}$$

i. Slow wave always less than intermediate wave  
Fast " " " " faster " " " "

Friedrich's diagram



as  $\frac{a^2}{b^2} \gg 1$  fast mode becomes ordinary sound wave

$$S_p = \frac{B_y \delta B_y}{4\pi} \frac{1}{u^2 - a^2} = \frac{1}{8\pi} \frac{\delta [B_y^2]}{u^2 - a^2}$$

$\therefore$  for fast wave  $u^2 > a^2$   
 and  $S_p, S_P, S(B_y^2)$  increase or decrease together

In slow wave  $u^2 < a^2$  and  $S(B_y^2)$  decreases when  $S_p$  and  $P$  increase and vice versa