

# Guiding Center motion

## Lecture 7

### Single Particle motion in $\vec{E}$ and $\vec{B}$ fields or Guiding Center motion

Equation of motion for a single particle

$$\star \quad \frac{d}{dt} m\vec{v} = e\vec{E} + e\vec{v} \times \vec{B}$$

First take  $\vec{E} = 0$  and  $\vec{B}$  arbitrary

Then  $\star$  becomes

positive for electrons

$$\frac{d}{dt} \vec{v} = \vec{\Omega} \times \vec{v} \quad \text{where } \vec{\Omega} = -\frac{e\vec{B}}{m}$$

Note  $|\vec{\Omega}| = \omega_c$  (Parks P. 1)

this represents a velocity vector instantaneously

rotating about  $\vec{B}$  with frequency  $\omega = \omega_c = \frac{eB}{m}$

i.e. The particle is instantaneously moving in an

arc of a circle, whose radius vector  $\vec{r}$  is defined by

$$\vec{v}_\perp = \vec{\Omega} \times \vec{r} \quad [\text{note } |\vec{r}| = \text{Parks } r_c]$$

where  $\vec{v}_\perp \Rightarrow$  component of  $\vec{v}$   $\perp$  to  $\vec{B}$ :

$$\begin{aligned} \vec{v}_\perp &\equiv \vec{v} - \vec{v}_{||} \quad \text{and} \quad \vec{v}_{||} \equiv \left( \vec{v} \cdot \frac{\vec{B}}{B} \right) \frac{\vec{B}}{B} \\ &= (\vec{v} \cdot \hat{B}) \hat{B} \end{aligned}$$

Now solve for  $\vec{r}$ :

$$\vec{r} \times \vec{v}_\perp = \vec{r} \times \vec{v} = \vec{r} \times (\vec{r} \times \vec{p}) =$$

$$\vec{r}(\vec{r} \cdot \vec{p}) - \vec{p} \cdot \vec{r}^2$$

but  $\vec{p}$  is  $\perp$  to  $\vec{B}$   $\therefore \vec{r} \cdot \vec{p} = 0$

$$\Rightarrow \vec{p} = \frac{\vec{v} \times \vec{r}}{\vec{r}^2} = \frac{m}{e} \frac{\vec{B} \times \vec{v}}{B^2} = \frac{\vec{B} \times \vec{p}}{e B^2}$$

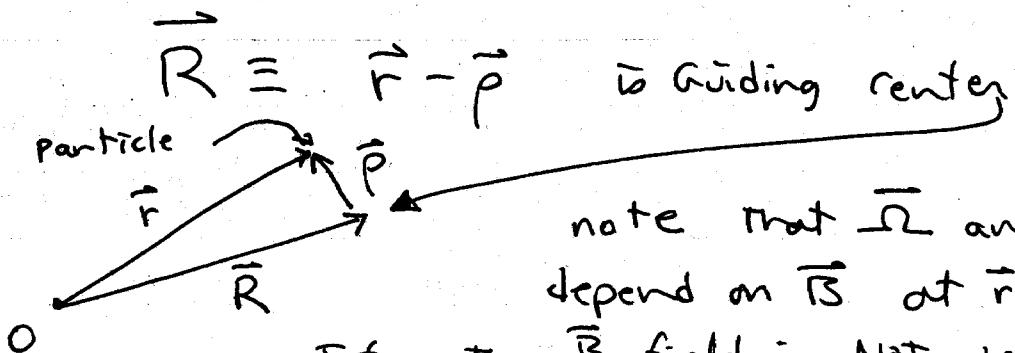
where  $\hat{p} = m\vec{v}$  = momentum

(not to be confused with  $\vec{p} \rightarrow$  radius of circle)

N.B. this  
definition of  $\vec{p}$   
is just opposite  
part

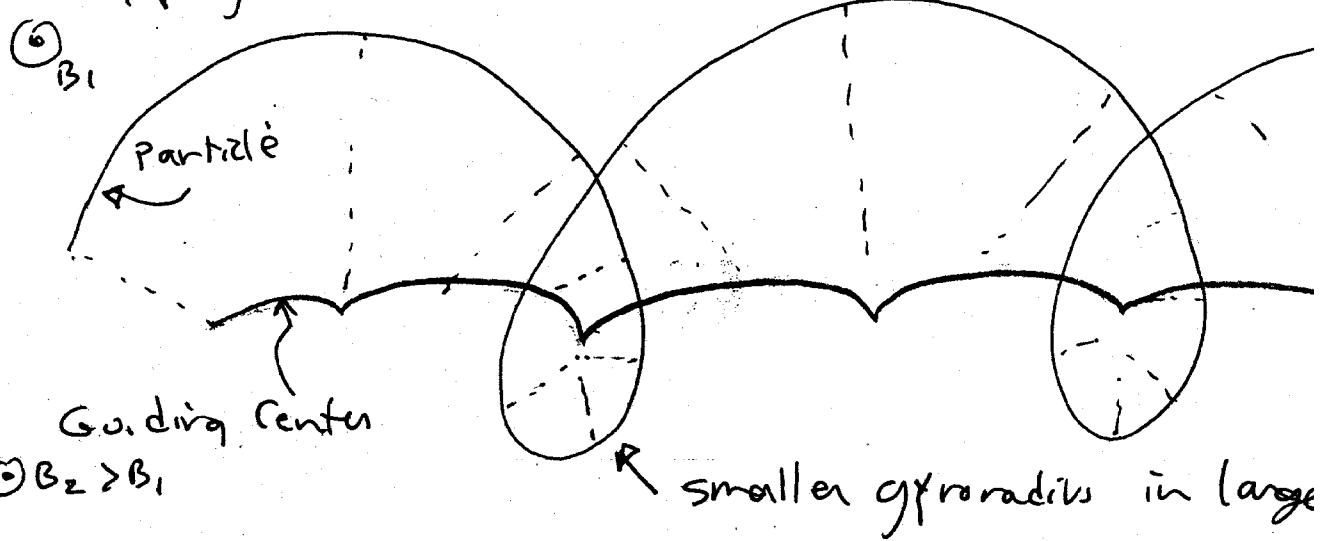
Position of the center of the circle is  
called the Guiding center. Note this

[def. of  $\vec{p}$  is valid ONLY in a frame where  $\vec{E}_\perp = 0$   
see this later]



If the  $\vec{B}$  field is NOT uniform, after  
the particle has moved a short arc of the circle,  
the frequency and radius will be different and  
Therefore the trajectory not a closed circle.

Example : Let  $\vec{B}$  be ~~into~~<sup>out of the</sup> page everywhere but increase in magnitude toward bottom of page



But, if we assume  $\vec{B}$  changes very little over a distance equal to the gyro radius

i.e. if  $\vec{B}(\vec{r} + \vec{p}) - \vec{B}(\vec{r}) \ll \vec{B}(\vec{r})$

or  $L \gg p$  where  $L$  is scale length

of field variations ( $\vec{B}(B) \propto \frac{B}{L}$ )

(true for most plasmas)

then Guiding center oscillations with much smaller amplitude than the particle

See Figures

Usually we don't care about the particle as distinct from its Guiding Center, nor oscillations of the Guiding Center.

so we will consider the motion of the Guiding Center averaged over the particle gyration. (Called the 1<sup>st</sup> order orbit Theory or the Alfvén approximation.)

### [Definition of Guiding Center in Presence of Electric Field]

If  $\vec{E}$  and  $\vec{B}$  are fields viewed from one frame of reference, and  $\vec{E}'$  and  $\vec{B}'$  are from another moving with velocity  $\vec{v}$  with respect to the first, then as was shown earlier, the Lorentz Transformation (see Part 2.6) is:

$$\vec{E}'_t = \gamma (\vec{E}_t + \vec{v} \times \vec{B}) \quad \vec{E}'_l = \vec{E}_l$$

$$\vec{B}'_t = \gamma (\vec{B}_t - \frac{\vec{v} \times \vec{E}}{c^2}) \quad \vec{B}'_l = \vec{B}_l$$

where  $\gamma = (1 - \frac{v^2}{c^2})^{-1/2}$  and  $t, l$  refer to:

$\vec{E}_t$  is component  $\perp$  to  $\vec{v}$  (transverse)  
 $\vec{E}_l$  " "  $\parallel$  to  $\vec{v}$  (longitudinal)

for Nonrelativistic motion  $E \ll cB$   
 (true in plasmas)

$$\vec{E}' = \vec{E} + \vec{V} \times \vec{B}$$

$$\vec{B}' = \vec{B}$$

$\therefore \vec{E}_\perp = 0$  in a frame of reference moving

with velocity  $\vec{V}_B = \frac{\vec{E} \times \vec{B}}{B^2}$

in this frame the local particle motion  $\perp$  to  $B$  is gyration.

In general  $\vec{V}_B$  varies from point to point,

$\therefore \vec{E}_\perp$  vanishes in different frames at different points.

Note  $\vec{B} \cdot \vec{V}_B = 0$

### Define Guiding Center in General

$$\vec{R} = \vec{r} - \vec{p} \quad \text{because must transform away } \vec{E}_\perp$$

$$\vec{p} = \frac{m}{e} \frac{\vec{B} \times (\vec{v} - \vec{V}_B)}{B^2} = \frac{m}{e} \frac{\vec{B} \times \vec{v}^*}{B^2} = \frac{\vec{B} \times \vec{p}^*}{e B^2}$$

$$\vec{v}^* = \vec{v} - \vec{V}_B \quad \vec{p}^* = \vec{p} - m \vec{V}_B$$

$$\text{Guiding Center Velocity } \vec{V}_G = \frac{d \vec{R}}{dt} = \frac{d \vec{r}}{dt} - \frac{d \vec{p}}{dt}$$

$$\vec{V}_G = \vec{v} \left( + \frac{d}{dt} \left( \frac{\vec{p}^* \times \vec{B}}{e B^2} \right) \right) = \vec{v} + \frac{d \vec{p}^*}{dt} \times \frac{\vec{B}}{e B^2} + \frac{\vec{p}^*}{e} \times \frac{d}{dt} \left( \frac{\vec{B}}{e B^2} \right)$$

or  $\Rightarrow$

$$\bar{V}_G = \bar{v} + \left[ \frac{d}{dt} (\bar{P} - m \bar{V}_B) \right] \times \frac{\bar{B}}{e B^2} + \frac{\bar{P}^*}{e} \times \frac{d}{dt} \left( \frac{\bar{B}}{B^2} \right)$$

$$\text{but } \frac{d \bar{P}}{dt} = e \bar{E} + e \bar{V} \times \bar{B}$$

note

$$\bar{v} + (\bar{v} + \bar{B}) \times \frac{\bar{B}}{B^2} = \frac{\bar{B}}{B} \frac{\bar{B}}{B} \cdot \bar{v} = \bar{v}_{||} \text{ as before}$$

also

$$\frac{d}{dt} \left( \frac{\bar{B}}{B^2} \right) = \underbrace{\left( \frac{\partial}{\partial t} + \bar{v} \cdot \nabla \right)}_{\text{chain rule} \rightarrow \text{since } \bar{B}(\bar{x}, t)} \frac{\bar{B}}{B^2} = \left[ \frac{\partial}{\partial t} + (\bar{V}_B + \bar{v}^*) \cdot \nabla \right] \frac{\bar{B}}{B}$$

∴ Guiding Center velocity =

$$\begin{aligned} \bar{V}_G &= \frac{\bar{B}}{B} \bar{v} \cdot \frac{\bar{B}}{B} + \bar{V}_B + \frac{m}{e} \frac{\bar{B}}{B^2} \times \left[ \frac{\partial}{\partial t} \bar{V}_B + (\bar{v}^* + \bar{V}_B) \right] \\ &\quad + \frac{\bar{P}^*}{e} \times \left[ \frac{\partial}{\partial t} \frac{\bar{B}}{B^2} + (\bar{v}^* + \bar{V}_B) \cdot \nabla \frac{\bar{B}}{B^2} \right] \end{aligned}$$

So far - NO Approximation. All Fields evaluated at point of particle  $\bar{r}$  (Not  $\bar{R}$ )

Next: assume small / slow variations  
in  $\bar{B}, \bar{E}$  then make Approximations.

Now comes a bunch of math.

basically we want  $\bar{V}_G$  as a function

a  $\bar{R}$  not  $\bar{r}$ . To do this we need  
 $B(\bar{R})$  not  $\bar{B}(r)$  etc.

so Taylor series expand assuming  
 $\bar{r} = R + p$  where  $p \ll R$

$$\text{so } B(\bar{r}) \stackrel{\sim}{=} B(\bar{R}) + \bar{p} \cdot \nabla_{\bar{R}} B(\bar{R}) + \dots$$

$$\text{and } p \nabla B(r) \stackrel{\sim}{=} \underbrace{p \nabla_{\bar{R}} B(\bar{R})}_{\text{ }} + \dots$$

already 1<sup>st</sup> order in smallness

$$\frac{m}{eB} \sim \frac{\partial \vec{B}}{\partial t} \approx -\frac{mV_0}{eB(R)} \frac{\partial}{\partial t} (B(R,t) + \dots)$$

Plug into  $\bar{V}_g$  equation ( see Supplemental Notes)

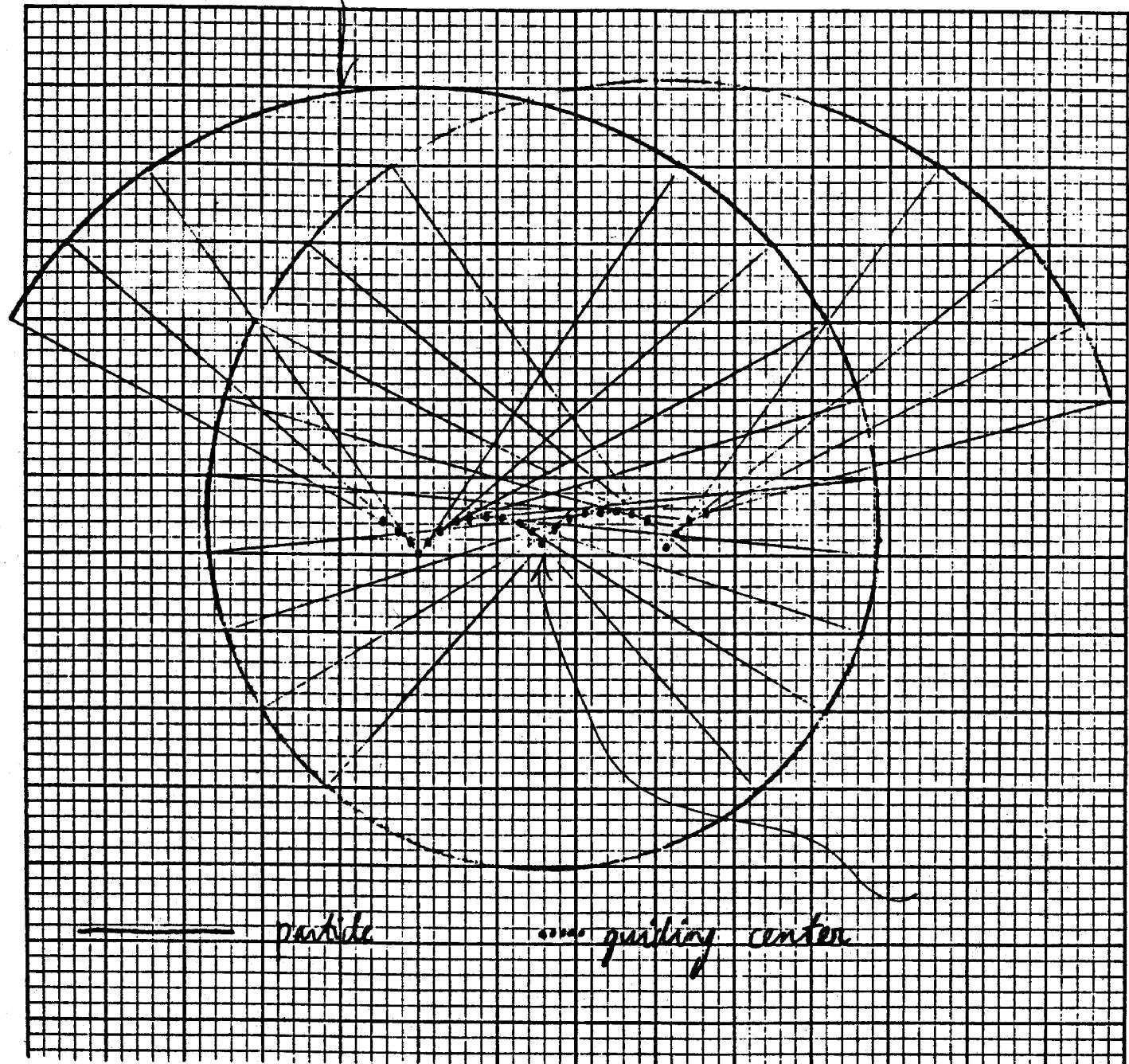
to get (to 1<sup>st</sup> order in smallness)

using Finite Larmor Radius ordering

$$\langle \bar{v}_\sigma \rangle = \bar{v}_{||} + \bar{v}_B + \frac{\bar{B}}{eB^2} \times \mu \bar{v} B + \frac{p_{||} v_{||}}{e} \underbrace{\frac{\bar{B} \times (\bar{B} \cdot \bar{\nabla}) \bar{B}}{B^4}}$$

ExB term ↑      Gradient Drift      Curvature Drift-

Particle



Alfvén and Fälthammar, Cosmical Electrodynamics  
 (Oxford, 1963)

§ 2.5

CHARGED PARTICLES IN MAGNETIC FIELDS

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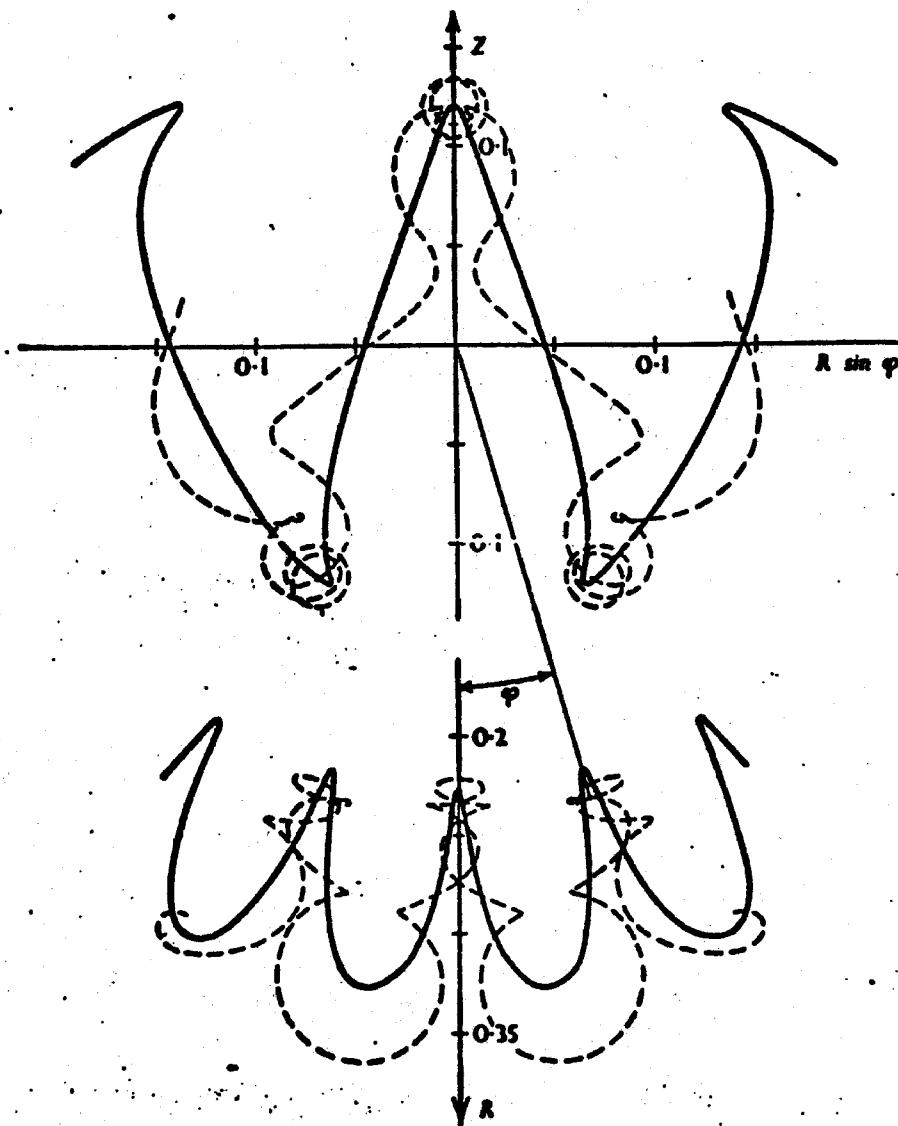


FIG. 2.9. Motion in dipole field calculated by Störmer and by the perturbation method. Upper figure: Projection upon a plane through the axis of the dipole. Lower figure: Projection upon the equatorial plane. — path of equivalent magnet. ····· path of the particle according to Störmer.