Supplemental Notes

Guiding Center Notes #2

1st (Review motion from Lorentz Force point of view)

then Assume all fields vary on space and time scales Large WRT ρ and $\frac{1}{\Omega}$ respectively

Let L = characteristic spatial scale for variations
τ = " temporal " " "

and let $\frac{\rho}{L} \sim \mathcal{O}(\epsilon)$ and $\frac{1}{\tau\Omega} \sim \mathcal{O}(\epsilon)$

Notation: ϵ is an ordering parameter
value of $\epsilon = 1$
Power of ϵ tells order of smallness
(eg. second order in smallness $\propto \mathcal{O}(\epsilon^2)$)

Now, also assume

$$\frac{V_D}{V} \sim \mathcal{O}(1) \qquad \frac{L}{V_G \tau} \sim \mathcal{O}(1)$$

and parallel motion in gyro period is small

$$\frac{v_\parallel}{L\Omega} \sim \mathcal{O}(\epsilon) \qquad \text{or} \quad \frac{v_\parallel}{L} \sim \epsilon\Omega$$

Note: this limits $E_\parallel$:

$$v_\parallel = \underset{\text{speed}}{\text{parallel}} \sim \underbrace{\frac{eE_\parallel}{m}}_{\text{acceleration}} \underbrace{\frac{L}{v_\parallel}}_{\text{time}} \qquad \therefore \frac{e}{m}\frac{E_\parallel}{v_\parallel} \lesssim \epsilon\Omega$$

So $\boxed{E_\parallel \lesssim \epsilon v_\parallel B}$

This ordering is less strict than IFLR

in the end, we will go to FLR ordering to reduce equations further.

GCN2 (Cont'd)

Particle motion $\vec{v}^* = \vec{v}_0^* + \epsilon \vec{V}_1 + \epsilon^2 \vec{V}_2 + \cdots$

where $\vec{v}_0^*$ motion in uniform field

$\vec{v}_0^* = \hat{x} v_\perp^* \cos \Omega t - \hat{y} V_\perp^* \sin \Omega t + \hat{z} v_\parallel$

(if B is in $\hat{z}$ direction)

Also $\vec{r} = \vec{R} + \epsilon \vec{\rho} + \cdots$

$$\boxed{\text{Expand } V_G \text{ to } 1^{st} \text{ order then Average over gyration}}$$

Taylor series expand $\vec{B}(r)$

we have

$$B(\vec{r}) = B(\vec{R}) + \epsilon(\vec{\rho}_0 \cdot \nabla_{\vec{R}}) \vec{B}(\vec{R}) + \cdots$$

and

$$\vec{\rho} \nabla \vec{B}(r) = \epsilon \rho_0 \nabla_{\vec{R}} \vec{B}(\vec{R}) + \cdots$$

$$\frac{m}{eB} v \frac{\partial \vec{B}}{\partial t} = \epsilon \frac{m v_0}{e B(\vec{R})} \frac{\partial}{\partial t} B(R,t) + \cdots$$

Then

$$\vec{V}_G = \frac{\vec{B}}{B} \vec{v} \cdot \frac{\vec{B}}{B} + \vec{V}_B + \epsilon \frac{m}{e} \frac{\vec{B}}{B^2} \times \left[\frac{\partial}{\partial t} \vec{V}_B + (\vec{V}_0^* + \vec{V}_B) \right.$$

$$+ \epsilon \frac{\vec{P}_0^*}{e} \times \left[\frac{\partial}{\partial t} \frac{\vec{B}}{B^2} + (\vec{V}_0^* + \vec{V}_B) \cdot \nabla \frac{\vec{B}}{B^2} \right] +$$

$$+ \epsilon \frac{\vec{B}}{B} (\vec{v}_0^* + \vec{V}_B) \cdot \left[\left(\frac{\vec{B} \times \vec{P}_0^*}{e B^2} \right) \cdot \nabla \right] \frac{\vec{B}}{B} + \epsilon \left(\vec{\rho}_0 \cdot \nabla \frac{\vec{B}}{B} \right) \vec{v}_0^* \cdot \frac{\vec{B}}{B}$$

$$+ \epsilon \vec{\rho}_0 \cdot \nabla \vec{V}_B$$

where ALL Fields are now evaluated at $\vec{R}$

Now Average $\bar{V}_G$ over gyration $\langle \bar{V}_G \rangle$
(drop subscript $_0$ on $\bar{p}^*$, $\bar{v}^*$, ρ)

$$\langle \bar{v}^* \rangle = v_{\parallel} \frac{\bar{B}}{B} \qquad \langle \bar{\rho} \rangle = 0$$

$$\langle \bar{p}^* v^* \rangle = \begin{pmatrix} \frac{1}{2} p_\perp^* v_\perp^* & 0 & 0 \\ 0 & \frac{1}{2} p_\perp^* v_\perp^* & 0 \\ 0 & 0 & p_\parallel v_\parallel \end{pmatrix}$$

if set $\hat{z} = \frac{\bar{B}}{B}$

$$= \frac{1}{2} p_\perp^* v_\perp^* \overleftrightarrow{1} + \left(p_\parallel v_\parallel - \frac{1}{2} p_\perp^* v_\perp^* \right) \frac{\bar{B}\bar{B}}{B^2}$$

(in general)

unit tensor $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\langle \bar{\rho}_0 \underbrace{\bar{v}_0^* \cdot B}_{\sim v_\parallel^*} \rangle = 0$$

<u>To obtain:</u>

$$\langle \bar{V}_G \rangle = v_\parallel \frac{\bar{B}}{B} + \bar{V}_B + \epsilon \frac{m}{e} \frac{\bar{B}}{B^2} \times \left[\frac{\partial}{\partial t} + \left(v_\parallel \frac{\bar{B}}{B} + V_B \right) \cdot \nabla \right] \bar{V}_B$$

$$+ \epsilon \frac{p_\parallel}{e} \frac{\bar{B}}{B} \times \left[\frac{\partial}{\partial t} + \left(v_\parallel \frac{\bar{B}}{B} + V_B \right) \cdot \nabla \right] \frac{\bar{B}}{B^2} +$$

$$+ \epsilon \frac{1}{\rho} \left\langle \bar{p}_\perp^* \times \frac{\bar{B}}{B} V_\perp^* \right\rangle \cdot \nabla \frac{1}{B} +$$

$$\epsilon \frac{1}{eB} \left\{ \left\langle \bar{p}_\perp^* \times (\bar{v}_\perp^* \cdot \nabla) \frac{\bar{B}}{B} \right\rangle + \frac{\bar{B}}{B} \left\langle \bar{v}_\perp^* \cdot \left[\left(\frac{\bar{B}}{B} \times \vec{p}_\perp^* \right) \cdot \nabla \right] \frac{\bar{B}}{B} \right\rangle \right.$$

Note that $v_\parallel = v \cdot \frac{\bar{B}(\bar{R})}{B(\bar{R})}$ Parallel to $\bar{B}$ at $\bar{R}$ Guiding center

GCN 2 (cont'd)

define $\left(\frac{d}{dt}\right)_0 \equiv \frac{\partial}{\partial t} + \left(v_{\parallel}\frac{\bar{B}}{B} + \bar{V}_B\right)\cdot\nabla$

time derivative following the zero-order motion to simplify $\langle \bar{V}_G \rangle$ &

2nd line: $\frac{\bar{B}}{B} \times \left(\frac{d}{dt}\right)_0 \frac{\bar{B}}{B^2} = \frac{\bar{B}}{B^2} \times \left(\frac{d}{dt}\right)_0 \frac{\bar{B}}{B} = \frac{\bar{B}}{B^3} \times \left(\frac{d}{dt}\right)_0 \bar{B}$

3rd line $\left\langle \bar{P}_\perp^* \times \frac{\bar{B}}{B} \bar{V}_\perp^* \right\rangle \cdot \nabla \frac{1}{B} = \frac{\bar{B}}{B^3} \times \langle \bar{P}_\perp^* \bar{v}_\perp^* \rangle \cdot \nabla B$

$= \frac{\bar{B}}{B^3} \times \frac{1}{2} P_\perp v_\perp \nabla B$

4th line: can show $\{\ \} = 0$

$\bar{v}_\perp^* \cdot \nabla \frac{\bar{B}}{B}$ perpendicular to $\bar{B}$ $\therefore \bar{P}_\perp \times (\bar{v}_\perp^* \cdot \nabla)\frac{\bar{B}}{B}$ is $\parallel$

$\{\ \} = \left\langle \frac{\bar{B}}{B}\left[\left(\frac{\bar{B}}{B} \times \bar{P}_\perp^*\right)\cdot(\bar{v}_\perp^* \cdot \nabla)\frac{\bar{B}}{B} + \bar{v}_\perp^* \cdot \left(\left(\frac{\bar{B}}{B} \times \bar{P}_\perp^*\right)\cdot\nabla\right)\frac{\bar{B}}{B}\right.\right.$

Letting $\hat{z} = \frac{\bar{B}}{B}$, $[\ \ ] = \frac{1}{2} P_\perp v_\perp \left[\frac{\partial}{\partial x}\frac{B_y}{B} - \frac{\partial}{\partial y}\frac{B_x}{B} + \frac{\partial}{\partial y}\frac{B_x}{B} - \frac{\partial}{\partial x}\right.$

$= \frac{1}{2} P_\perp v_\perp \frac{\bar{B}}{B} \cdot \left[\nabla \times \frac{\bar{B}}{B} - \nabla \times \frac{\bar{B}}{B}\right] = 0$ in Gen

Note that the two terms may cancel – but they are not zero individually when $\bar{B}\cdot\left(\nabla \times \frac{\bar{B}}{B}\right) = \frac{\bar{B}}{B}\cdot(\nabla \times$

GCN2 (cont'd)

thus (finally!), dropping ϵ and rearranging terms

$$\langle \bar{V}_G \rangle = \bar{v}_{\parallel} + \bar{V}_B + \frac{1}{e}\frac{\bar{B}}{B^2} \times \left[\mu \bar{\nabla} B + m v_{\parallel} \left(\frac{d}{dt}\right)_0 \frac{\bar{B}}{B} + \left(\frac{d}{dt}\right)_0 m \bar{V}_B \right]$$

where $\mu \equiv \frac{1}{2} \frac{P_\perp v_\perp}{B}$

1st line = zero order, 2nd line 1st order

we Assumed ordering $\frac{V_B}{V} \sim \mathcal{O}(1)$ and found $E_{\parallel} \sim \epsilon v B$

so $\frac{E_{\parallel}}{E_\perp} \sim \mathcal{O}(\epsilon)$

FLR (Finite Larmor Radius) ORDERING

$\frac{V_B}{V} \sim \mathcal{O}(\epsilon)$ so $\frac{V_{G\perp}}{v} \sim \mathcal{O}(\epsilon)$

$\frac{1}{\Omega \tau} \sim \mathcal{O}(\epsilon^2)$ $\frac{E_{\parallel}}{E_\perp} \sim \mathcal{O}(\epsilon^2)$

Then to 1st order in ϵ (the time derivatives drop out)

$$\langle \bar{V}_G \rangle = \bar{v}_{\parallel} + \bar{V}_B + \underbrace{\frac{\bar{B}}{eB^2} \times \mu \bar{\nabla} B}_{\text{Gradient Drift}} + \underbrace{\frac{P_{\parallel} v_{\parallel}}{e} \frac{\bar{B} \times (\bar{B} \cdot \nabla) \bar{B}}{B^4}}_{\text{Curvature Drift}}$$

no assumption on $v_{\parallel}$ this ordering ($E_{\parallel}$ zero, not $v_{\parallel}$)