Parallel motion of the Guiding Center

$$\langle V_{G\parallel} \rangle = \langle \vec{v} \rangle \cdot \frac{\vec{B}(R)}{B(R)}$$

Look at $\frac{d}{dt} m \vec{v} \cdot \frac{\vec{B}}{B} \Big|_R = m\vec{v} \cdot \frac{d}{dt}\frac{\vec{B}}{B} + \frac{\vec{B}}{B} \cdot \frac{d}{dt} m\vec{v}$

but Lorentz Force $= \frac{d}{dt} m\vec{v} = e\vec{E}(\vec{r}) + e\vec{v} \times \vec{B}(\vec{r})$

$$= e\vec{E}(\vec{R}) + e\vec{v} \times \vec{B}(R) + \epsilon\left\{ e\vec{\rho}\cdot\vec{\nabla}E + e\vec{v} \times (\vec{\rho}\cdot\vec{\nabla})\vec{B} \right\} + \dots \qquad \vec{\rho} = \frac{\vec{B}\times}{qB}$$

$$\therefore \frac{d}{dt} m\vec{v} \cdot \frac{\vec{B}}{B} = m\vec{v} \cdot \left(\frac{\partial}{\partial t} + V_G \cdot \nabla\right)\frac{\vec{B}}{B} + \underbrace{\frac{\vec{B}}{B} \cdot e\vec{E}}_{=E_\parallel} +$$

$$\left(\frac{\vec{B}}{B} \times e\vec{v}^*\right) \cdot \left(\frac{\vec{B}\times m\vec{v}^*}{eB^2} \cdot \nabla\right)\vec{B} + \text{Terms linear in } \rho$$

Now average over gyration $\vec{v}$ to 1st order

$$\left\langle \frac{d}{dt} m V_{G\parallel} \right\rangle = eE_\parallel + m\underbrace{\left(\vec{V}_B + \frac{\vec{B}}{B} V_\parallel\right)}\cdot\left(\frac{\partial}{\partial t} + \vec{V}_G \cdot \nabla\right)\frac{\vec{B}}{B} +$$
$$\left\langle \frac{1}{B}\left(\frac{\vec{B}}{B} \times \vec{v}^*\right) \cdot \left(\frac{\vec{B}}{B} \times \vec{\rho}^* \cdot \nabla\right)\vec{B} \right\rangle$$

Last term =

$$\frac{1}{2}\frac{\rho_\perp v_\perp}{B}\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}\right)\vec{B} = \mu\left(\cancel{\vec{\nabla}\cdot\vec{B}}^{\,0} - \frac{\partial B_z}{\partial z}\right) =$$

$$= -\mu\frac{\partial B_z}{\partial z} \qquad \text{if } \frac{\vec{B}}{B} = \hat{z}$$

$$= -\mu\frac{\vec{B}}{B} \cdot \vec{\nabla}B \qquad \text{in general}$$

$$\mu \equiv \frac{\frac{1}{2}P_\perp V_\perp}{B}$$

(over)

also to 1st order in ϵ: $(\bar{V}_G \cdot \nabla) = ((\bar{V}_B + \bar{v}_{\parallel}) \cdot \nabla)$

$$\therefore \left\langle \frac{d}{dt} m V_{G_{\parallel}} \right\rangle = \frac{d}{dt} m v_{\parallel} = e E_{\parallel} - \mu \left(\frac{\bar{B}}{B} \cdot \bar{\nabla} \right) B + m \bar{V}_B \cdot \left(\frac{d}{dt} \right)_0 \frac{\bar{B}}{B}$$

or, with FLR ordering **Finite Larmor Rad ordu**

$$\boxed{\frac{d}{dt} m v_{\parallel} = e E_{\parallel} - \mu \frac{\partial B}{\partial l}}$$

where $\frac{\partial}{\partial l} \equiv \frac{\bar{B}}{B} \cdot \bar{\nabla}$

Parallel Force equation

ONE can easily show μ = constant for $E_{\parallel} = 0$

(Proof): $m \frac{dv_{\parallel}}{dt} = e \cancel{E_{\parallel}}^{0} - \mu \frac{\partial B}{\partial l}$

multiply by $v_{\parallel} = \frac{\partial l}{\partial t}$

① $$\frac{d}{dt}\left(\tfrac{1}{2} m v_{\parallel}^2\right) = -\mu \cancel{\frac{\partial B}{\partial l}} \underbrace{\frac{\partial l}{\partial t}}_{v_{\parallel}} = -\mu \frac{\partial B}{\partial t}$$

Energy must be conserved so

② $$\frac{d}{dt}\left(\tfrac{1}{2} m v_{\parallel}^2 + \tfrac{1}{2} m v_{\perp}^2\right) = \frac{d}{dt}\left(\tfrac{1}{2} m v_{\parallel}^2 + \mu B\right) = 0$$

$\mu \equiv \frac{\frac{1}{2} m v_{\perp}^2}{B}$

so, combining ① + ②

$$-\mu \frac{dB}{dt} + \frac{d}{dt}(\mu B) = 0 \quad \text{or} \quad \boxed{\frac{d\mu}{dt} = 0}$$

$\Rightarrow$ μ = constant of motion when energy is conserved on times of the gyration period

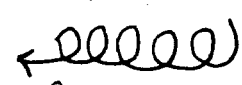
Physical Interpretation of terms of Guiding Center Equations.

$$\vec{V}_G = \vec{v}_{\parallel} + \underbrace{\frac{\bar{E}\times\bar{B}}{B^2}}_{\vec{V}_B} + \frac{\vec{B}}{eB^2}\times\mu\bar{\nabla}B + \frac{\vec{B}}{eB^2}\times m v_{\parallel}^2\,\frac{(\bar{B}\cdot\bar{\nabla})\bar{B}}{B^2}$$

Consider $\frac{\bar{E}\times\bar{B}}{B^2}$ term $(\vec{V}_B)$

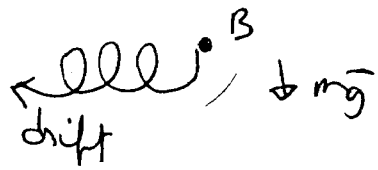
this term is of the form $\frac{\bar{F}\times\bar{B}}{eB^2}$ where $\bar{F} = e\bar{E}$

The drift velocity is independent of the form of the force and would be the same for any force $\vec{F}$ eg $B\bullet \downarrow \vec{F}$ causes particle to move



thus, for example a gravitational field

$$\frac{m\vec{g}\times\bar{B}}{eB^2}$$



The $\frac{-\mu\bar{\nabla}B\times\bar{B}}{B^2}$ term

Drift due to force on magnetic moment due to the gradient in the magnetic field

Energy $= \mu B$ $\qquad \mu$ = constant

Force $= -\nabla(\mu B) = -\mu\bar{\nabla}B$

Called <u>Gradient Drift term</u>

The $-\frac{mv_{\parallel}^2}{e}\frac{(\bar{B}\cdot\nabla\bar{B})\bar{B}}{B^2}\times\frac{\bar{B}}{B^2}$ term

This results from the centrifugal force $\left(\frac{-mv_{\parallel}^2}{R_c}\right)$ and is called the <u>Curvature Drift</u>

It results from centrifugal force felt by particle moving along curved $\vec{B}$

Note when $\bar{\nabla}\times\bar{B}=0$ we can evaluate Curvature drift in a coordinate system with $\bar{B}$ along $\hat{z}$

$$(\bar{B}\cdot\bar{\nabla})\bar{B} = B_z\frac{\partial\bar{B}}{\partial z}$$

we want the component of this function which is $\perp B$ since it gets crossed with $\bar{B}$ to give the drift

This component is $\hat{x}\,B_z\frac{\partial B_x}{\partial z}+\hat{y}\,B_z\frac{\partial B_y}{\partial z}$

from $\bar{\nabla}\times\bar{B}=0\ (=J)$

$$\left.\begin{aligned}\frac{\partial B_x}{\partial z} &= +\frac{\partial B_z}{\partial x}\\ \frac{\partial B_y}{\partial z} &= +\frac{\partial B_z}{\partial y}\end{aligned}\right\}\Rightarrow \begin{aligned}&\hat{x}\,B_z\frac{\partial B_x}{\partial z}+\hat{y}\,B_z\frac{\partial B_y}{\partial z}\\ &=\hat{x}\,B_z\frac{\partial B_z}{\partial x}+\hat{y}\,B_z\frac{\partial B_z}{\partial y}\\ &=B_z\bar{\nabla}B\end{aligned}$$

So we get

$$\text{Curvature drift} = -\frac{mv_{\parallel}^2}{e}\frac{(\bar{B}\cdot\nabla)\bar{B}}{B^2}\times\frac{\bar{B}}{B^2} = -\frac{mv_{\parallel}^2}{e}\frac{1}{B^3}\frac{\partial\bar{B}}{\partial z}\times\bar{B}$$

$$= -\frac{mv_{\parallel}^2}{eB^3}\vec{\nabla}B\times\bar{B}$$

also see Parks (4.78)

So we can combine ∇B and Curvature B drifts when $J = 0 = \vec{\nabla} \times \vec{B}$

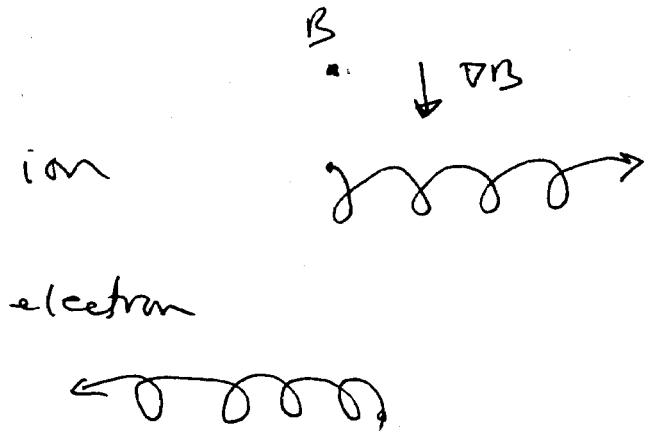
$$W_c + W_{\nabla B} = \text{Curvature} + \text{Gradient Drifts} =$$

$$= \frac{e}{B^3} (\vec{B} \times \vec{\nabla} B)\left(\frac{m v_\perp^2}{2} + m v_\parallel^2 \right)$$

$$= \frac{e}{B^3} (\vec{B} \times \vec{\nabla} B)\left(\mathcal{E}_\perp + 2\mathcal{E}_\parallel \right)$$

where $\mathcal{E}_\perp \equiv \frac{1}{2} m v_\perp^2$ and $\mathcal{E}_\parallel = \frac{1}{2} m v_\parallel^2$

$\boxed{\text{Note}}$ Currents caused by Grad and Curvature drifts



$\therefore$ ions + electrons drift in opposite directions and <u>produce a current</u>

Can you calculate current in some complicated plasma problem this way? NO why not?