

Formal Derivation of Macroscopic Plasma Equations

Boltzmann equation (B'eqn)

$$\frac{\partial f_a}{\partial t} + \nabla \cdot \bar{v} f_a + \nabla_v \cdot \left\{ \left[\frac{z_a e}{m_a} (\bar{E} + \nabla \times \bar{B}) + \bar{g} \right] f_a \right\} = \frac{\delta f_a}{\delta t}_c$$

where $\left. \frac{\delta f}{\delta t} \right)_c$ symbolically represents the change in f_a due to collisions

> to obtain an equation for $\rho = \sum_a m_a \int d^3v f_a$
 multiply B'eqn by m_a , integrate over velocity space, and sum over all species

> to obtain an equation for $\rho \bar{v} = \sum_a m_a \int d^3v \bar{v} f_a$
 multiply B'eqn by $m_a \bar{v}$, integrate over velocity space, and sum over all species
 etc
 :

Formally taking moments of the distribution function

Continuity Eqn

consider each term separately

$$\sum_a m_a \int d^3v \frac{\partial f_a}{\partial t} = \frac{\partial}{\partial t} \sum_a m_a \int d^3v f_a = \frac{\partial \rho}{\partial t}$$

$$\sum_a m_a \int d^3v \nabla \cdot \vec{v} f_a = \nabla \cdot \sum_a m_a \int d^3v \vec{v} f_a = \nabla \cdot \rho \vec{V}$$

write $\vec{F}_a \equiv z_a e (\vec{E} + \vec{v} \times \vec{B}) + m_a \vec{g}$

then

$$\int d^3v \nabla_v \cdot \vec{F}_a f_a = \oint_{v \rightarrow \infty} d\vec{S}_v \cdot \vec{F}_a f_a = 0$$

where the second integral is a surface integral in velocity space (use Divergence Theorem) but the surface is at ∞ where f_a vanishes.

~~$\int d^3v \frac{\delta f}{\delta t} = 0$~~ since collisions cannot change the total density of particles

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} = 0$$

continuity equation

Next: Momentum Eqn

Formal Derivation of equation for $\rho \bar{V}$ (Momentum Equation)

multiply Boinan eqn by $m_a \bar{v}$, integrate over velocity space and sum over species

Term by term:

$$\sum_a m_a \int d^3v \bar{v} \frac{\partial f_a}{\partial t} = \frac{\partial}{\partial t} \rho \bar{V}$$

see last lecture ↓

$$\sum_a m_a \int d^3v \bar{v} \bar{v} \cdot \nabla f_a = \nabla \cdot \sum_a m_a \int d^3v \bar{v} \bar{v} f_a = \nabla \cdot \bar{K} =$$

$$= \nabla \cdot \sum_a m_a \int d^3v (\bar{v}^* + \bar{V})(\bar{v}^* + \bar{V}) f_a =$$

$$= \nabla \cdot \sum_a m_a \int d^3v (\bar{v}^* \bar{v}^* + \bar{V} \bar{v}^* + \bar{v}^* \bar{V} + \bar{V} \bar{V}) f_a$$

$$\text{but } \sum_a m_a \int d^3v \bar{v}^* f_a = \sum_a m_a \int d^3v (\bar{v} - \bar{V}) f_a =$$

$$\sum_a m_a \int d^3v \bar{v} f_a - \bar{V} \sum_a m_a \int d^3v f_a = \rho \bar{V} - \rho \bar{V} = 0$$

so middle two terms drop out

$$\therefore \nabla \cdot \bar{K} = \nabla \cdot (\bar{P} + \rho \bar{V} \bar{V})$$

$$\bar{v} \bar{\nabla}_v \cdot \bar{F}_a f_a = \bar{\nabla}_v \cdot (\bar{F}_a \bar{v} f_a) - f \bar{F}_a \cdot \bar{\nabla}_v \bar{v} = \bar{\nabla}_v \cdot (\bar{F}_a \bar{v} f_a) - f_a \bar{F}_a$$

or, in component form



$$v_i \frac{\partial}{\partial v_j} F_{aj} f_a = \frac{\partial}{\partial v_j} (v_i F_{aj} f_a) - F_{aj} f_a \frac{\partial v_i}{\partial v_j}$$

$$\text{but } F_{aj} \frac{\partial v_i}{\partial v_j} = F_{aj} \delta_{ij} = F_{ai}$$

$$\therefore \sum_a m_a \int d^3v \bar{v} \nabla_i \frac{F_a}{m_a} f = - \sum_a \int d^3v \bar{F}_a f_a$$

where the 1st term $\nabla_i \cdot (\bar{F}_a \bar{v} f_a)$ gives zero on integration, becoming a surface integral at ∞ where $f_a \rightarrow 0$

$$\sum_a \int d^3v \bar{F}_a f = \rho_c \bar{E} + \bar{J} \times \bar{B} + \rho \bar{g}$$

where $\rho_c = \sum_a Z_a e \int d^3v f_a$ Charge Density

$$\sum_a m_a \int d^3v \bar{v} \left(\frac{\delta f_a}{\delta t} \right)_c = 0 \quad \text{since momentum is conserved in collisions}$$

\therefore collisions cannot change total momentum density of plasma. This is true ONLY if we sum over all the species. (collisions can change momentum of one species relative to another.)

compare to Pines Eqn 5.35 p.157

$$\frac{\partial}{\partial t} \rho \bar{v} + \nabla \cdot (\rho \bar{v} \bar{v} + \bar{P}) = \rho_c \bar{E} + \bar{J} \times \bar{B} + \rho \bar{g}$$

Momentum Equation (Plasma counterpart to $\bar{F} = m\bar{a}$)

1st term = Time derivative of momentum density

2nd term = Divergence of momentum flux density

\therefore LHS = total rate of change of momentum density.

RHS = Force / unit volume

Kinetic Energy Density

$$\sum_a \frac{1}{2} m_a \int d^3v v^2 f_a = \sum_a \frac{1}{2} m_a \int d^3v (\vec{v}^* + \vec{V}) \cdot (\vec{v}^* + \vec{V}) f_a$$

$$= U_T + \frac{1}{2} \rho V^2$$

where $U_T = \sum_a \frac{1}{2} m_a \int d^3v v^{*2} f_a$ internal "thermal" energy density
 $\frac{1}{2} \rho V^2 =$ energy density associated with bulk motion of plasmas

U_T is related to pressure tensor:

$$U_T = \sum_a \frac{1}{2} m_a \int d^3v (v_x^{*2} + v_y^{*2} + v_z^{*2}) f_a =$$

$$= \frac{1}{2} (P_{xx} + P_{yy} + P_{zz}) = \frac{1}{2} \text{Trace}(\hat{P}) = \frac{1}{2} P_{ii}$$

Trace of a tensor = sum of diagonal elements

For isotropic pressure $P_{xx} = P_{yy} = P_{zz} = P$

and $U_T = \frac{3}{2} P$

For axially symmetric ("gyrotropic") pressure

with $\hat{z} = \hat{B}$

$P_{xx} = P_{yy} = P_{\perp}$; $P_{zz} = P_{\parallel}$

and $U_T = P_{\perp} + \frac{1}{2} P_{\parallel}$

(consider each term as before)

Equation for Kinetic Energy Density

$$\sum_a \frac{1}{2} m_a \int d^3v v^2 \frac{\partial f_a}{\partial t} = \frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 + U_T \right)$$

$$\sum_a \frac{1}{2} m_a \int d^3v v^2 \bar{\nabla} \cdot \bar{v} f_a = \bar{\nabla} \cdot \left\{ \sum_a \frac{1}{2} m_a \int d^3v \bar{v} v^2 f_a \right\}$$

$$\text{but } \bar{v} v^2 = (\bar{v}^x + \bar{V}) (v^{x^2} + 2\bar{V} \cdot \bar{v}^x + v^2) =$$

$$= \bar{V} v^2 + 2\bar{V} \bar{V} \cdot \bar{v}^x + v^2 \bar{v}^x + \bar{V} v^{x^2} + 2\bar{V} \cdot \bar{v}^x \bar{v}^x + \bar{v}^x v^{x^2}$$

$$\therefore \bar{\nabla} \cdot \left\{ \right\} = \bar{\nabla} \cdot \left[\frac{1}{2} \rho v^2 \bar{V} + \bar{V} U_T + \bar{V} \cdot \bar{P} + \bar{q} \right]$$

(Terms linear in \bar{v}^x give zero, as on page 1)

$$\text{where } \bar{q} = \sum_a \frac{1}{2} m_a \int d^3v \bar{v}^x v^{x^2} f_a \equiv \text{heat Flux}$$

(Transport of internal energy by internal motion)

Last term LHS:

$$\frac{1}{2} v^2 \bar{\nabla}_v \cdot \bar{F}_a f_a = \bar{\nabla}_v \cdot \left(\frac{1}{2} \bar{F}_a f_a v^2 \right) - \frac{1}{2} f_a \bar{F}_a \cdot \bar{\nabla}_v v^2 =$$

$$= \bar{\nabla}_v \cdot \left(\frac{1}{2} \bar{F}_a f_a v^2 \right) - f_a \bar{F}_a \cdot \bar{v}$$

$$\therefore \sum_a \frac{1}{2} m_a \int d^3v v^2 \bar{\nabla}_v \cdot \frac{\bar{F}_a}{m_a} f_a = - \sum_a \int d^3v \bar{F}_a \cdot \bar{v} f_a =$$

$$= - \bar{E} \cdot \bar{J} - \rho \bar{V} \cdot \bar{q}$$

RHS:

$\sum_a \frac{1}{2} m_a \int d^3v v^2 \left(\frac{\delta f_a}{\delta t} \right)_c = 0$ if collisions are elastic, then kinetic energy is conserved in collisions and collisions cannot change total energy density (summed over species) of the plasma.

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 + U_T \right) + \nabla \cdot \left(\frac{1}{2} \rho v^2 \bar{v} + U_T \bar{v} + \bar{P} \cdot \bar{v} + \bar{q} \right) = \bar{E} \cdot \bar{J} + \rho \bar{v} \cdot \bar{g}$$

Energy Equation: Time rate of change of kinetic energy density (sum of bulk motion and internal energy), plus divergence of energy flux density (bulk energy carried by bulk motion, internal energy carried by bulk motion, work done by pressure and heat flux (internal energy carried by internal motions), equals work per unit volume done by electromagnetic and gravitational forces.

Summation of Single Particle Guiding Center Motions to obtain Electrical Current

In FLR ordering (where $v_B \sim \epsilon v$) ^{all drifts are small}
we found Guiding center drift

$$\vec{V}_G = \vec{v}_{||} + \frac{\vec{E} \times \vec{B}}{B^2} + \frac{1}{2} \frac{P_{\perp} v_{\perp}}{z_a e} \frac{\vec{B} \times \nabla B}{B^3} + \frac{P_{||} v_{||}}{z_a e} \frac{\vec{B} \times (\vec{B} \cdot \nabla) \vec{B}}{B^4}$$

we defined $\vec{\mu} =$ particle magnetic moment $= -\frac{E_{\perp}}{B} \frac{\vec{B}}{B} = -\frac{1}{2} \frac{P_{\perp} v_{\perp}}{B^2} \vec{B}$

The current due to particle drifts is then given by

$$\vec{J}_G = \sum_a z_a e \int d^3v f_a \vec{V}_G$$

We also used Magnetization Current

$$\vec{J}_M = \nabla \times \vec{M} \quad \text{where} \quad \vec{M} = \sum_a \int d^3v f_a \vec{\mu}$$

Look at Perpendicular Drift Current

$$\vec{J}_{G\perp} = \frac{\vec{B} \times \nabla B}{B^3} P_{\perp} + \frac{\vec{B} \times (\vec{B} \cdot \nabla) \vec{B}}{B^4} P_{||}$$

and Magnetization Current $\vec{J}_M = -\nabla \times \left(P_{\perp} \frac{\vec{B}}{B^2} \right)$

Important Vector identity: $\vec{B} \times (\nabla \times \vec{B}) = \vec{B} \nabla \cdot \vec{B} - \underbrace{(\vec{B} \cdot \nabla) \vec{B}}_{\text{magnitude } B}$

$$\therefore (\nabla \times \vec{B})_{\perp} = \vec{B} \times \left\{ \frac{(\vec{B} \cdot \nabla) \vec{B}}{B^2} - \frac{\nabla B}{B} \right\}$$

Perpendicular Magnetization Current

$$\vec{J}_{M_{\perp}} = \frac{\vec{B}}{B^2} \times \nabla P_{\perp} - P_{\perp} \frac{(\nabla \times \vec{B})_{\perp}}{B^2} - 2 P_{\perp} \frac{\vec{B} \times \nabla B}{B^3}$$

$$= \frac{\vec{B}}{B^2} \times \nabla P_{\perp} - P_{\perp} \left[\frac{\vec{B}}{B^3} \times \nabla B + \frac{\vec{B}}{B^2} \times (\vec{B} \cdot \nabla) \vec{B} \right]$$

total Perpendicular Current $\left\{ \begin{array}{l} \text{when } J_{M_{\perp}} \text{ \& } J_{O_{\perp}} \text{ added,} \\ \text{this term is important} \end{array} \right.$

$$\vec{J}_{\perp} = \vec{J}_{O_{\perp}} + \vec{J}_{M_{\perp}} = \frac{\vec{B}}{B^2} \times \nabla P_{\perp} + \frac{P_{\parallel} - P_{\perp}}{B^2} \vec{B} \times (\vec{B} \cdot \nabla) \vec{B}$$

$$\text{or } \boxed{\vec{J}_{\perp} = \frac{\vec{B}}{B^2} \times \left[\nabla P_{\perp} + \frac{P_{\parallel} - P_{\perp}}{B^2} (\vec{B} \cdot \nabla) \vec{B} \right]}$$

take Cross product with \vec{B} to compare to

which we will talk about

→ momentum equation term

$$\vec{J} \times \vec{B} = (\nabla P_{\perp})_{\perp} + \frac{P_{\parallel} - P_{\perp}}{B^2} \left[(\vec{B} \cdot \nabla) \vec{B} \right]_{\perp}$$

IF special case of isotropic pressure $P_{\parallel} = P_{\perp} \equiv P$

This reduces to

$$\boxed{\vec{J} \times \vec{B} = \nabla P}$$

!!

in general case $\vec{J} \times \vec{B} = \nabla \cdot \vec{P}$

Therefore There is No current unless There is a pressure gradient.

in case \vec{P} is uniform and \vec{B} is NOT uniform
 the drift current is exactly cancelled
 by the magnetization current!
 (Hence, our earlier statement about
 determining current from guiding center terms alone.)

For Completeness

$$\begin{aligned}
 \text{Parallel Current } \vec{j}_{\parallel} &= \sum_a z_a e \int d^3v f_a v_{\parallel} + \vec{\nabla} \times \vec{M} \cdot \frac{\vec{B}}{B} \\
 &= \sum_a z_a e \int d^3v f_a v_{\parallel} - \frac{P_{\perp}}{B} + \frac{B \cdot (\nabla \times B)}{B^2} = \\
 &= \sum_a z_a e \int d^3v f_a \left\{ v_{\parallel} - \frac{1}{2} \frac{P_{\perp}}{B} \frac{\vec{B}}{B} \cdot \nabla \times \frac{\vec{B}}{B} \right\} \\
 & \qquad \qquad \qquad = v_{\parallel}
 \end{aligned}$$

particle velocity, parallel to \vec{B} at the position
 of the particle (while v_{\parallel} is parallel to \vec{B}
 at position of guiding center)

$$\therefore \boxed{\vec{j}_{\parallel} = \sum_a z_a e \int d^3v f_a v_{\parallel}} \quad \text{true, but not helpful!}$$