

## Waves in Plasmas

Diellectric tensor approach

From Maxwell's equations, assuming plane waves:

$$i\bar{k} \times \bar{E} = i\omega \mu_0 \bar{H}$$

$$i\bar{k} \times \bar{H} = -i\omega \epsilon_0 \bar{E} + \bar{J}$$

$$i\bar{k} \cdot \bar{E} = \frac{\rho}{\epsilon_0}$$

$$i\bar{k} \cdot \bar{B} = 0$$

Assume  $\bar{J} = \sigma \cdot \bar{E}$   
Ohm's Law

Continuity equation :  $i\bar{k} \cdot \bar{J} = i\omega \rho_e$

$$\rho_e = \frac{\bar{k} \cdot \bar{J}}{\omega} = \frac{\bar{k} \cdot \sigma \cdot \bar{E}}{\omega}$$

$$\text{so } i\bar{k} \cdot \bar{E} = \frac{\bar{k} \cdot \sigma \cdot \bar{E}}{\omega \epsilon_0} \quad \text{or} \quad \bar{k} \cdot \left( \frac{i(\bar{E} - \bar{\sigma} \cdot \bar{E})}{\omega \epsilon_0} \right) = 0$$

Now let  $\bar{E} = \hat{I} \cdot \bar{E}$  and write ( $\hat{I}$  = unit-tensor)

$$i\bar{k} \cdot \bar{K} \cdot \bar{E} = 0 \quad \text{where } \bar{K} = \hat{I} - \frac{\sigma}{i\omega \epsilon_0}$$

is the dielectric tensor

with this definition we can use Maxwell's "curl" eqtns.

$$i\bar{k} \times \bar{H} = -(i\omega \epsilon_0 \bar{K} \cdot \bar{E}) \quad \text{then take } \bar{k} \text{ cross Faraday's law}$$

$$i\bar{k} \times (\bar{k} \times \bar{E}) = i\omega \mu_0 \bar{K} \times \bar{H} = -i\omega \mu_0 (i\omega \epsilon_0 \bar{K} \cdot \bar{E})$$

So  $\boxed{\bar{k} \times (\bar{k} \times \bar{E}) + \frac{\omega^2}{c^2} \bar{K} \cdot \bar{E} = 0}$  wave eqtn

Now we want to find an explicit form for  $\frac{d\vec{v}}{dt}$   
 Assume a steady, large magnetic field  $\vec{B}_0 = B_0 \hat{z}$   
 and a cold ( $T=0$ ) plasma

$$\text{Force Eqn: } \frac{d\vec{v}}{dt} = \frac{e}{m} \vec{E} + \frac{e}{m} \vec{v} \times \vec{B}$$

now linearize,  $\Rightarrow$  Note:

$$A: \frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \underbrace{\vec{v} \cdot \nabla \vec{v}}_{\text{second order}} \quad \text{but } \vec{v} = \vec{v}_0 + \vec{v}_i \underset{\text{small}}{\sim} \vec{v}_i$$

$$\frac{d\vec{v}}{dt} \underset{\text{in smallness}}{\sim} \frac{\partial \vec{v}}{\partial t}$$

$$B: \nabla \times \vec{B} \approx \vec{v} \times \vec{B}_0$$

$$C: \vec{j} = \rho \vec{v} \approx \rho v_i \quad \text{same assumptions}$$

in rectangular coordinates we will split up  
 the force equation in components:

$$(\text{Force Eqn}) \quad -i\omega \vec{v} = \frac{e}{m} \vec{E} - \vec{v} \times \vec{w}_c \quad \text{where } \vec{w}_c = \frac{-e \vec{B}_0}{m}$$

components:

$$-i\omega v_x = \frac{e}{m} E_x - w_c v_y$$

$$-i\omega v_y = \frac{e}{m} E_y + w_c v_x$$

$$-i\omega v_z = \frac{e}{m} E_z$$

$$\text{so } -i\omega v_x = \frac{e}{m} E_x - w_c \left( \frac{\frac{e}{m} E_y + w_c v_x}{-i\omega} \right)$$

rewriting  $\rightarrow$

$$V_x(\omega^2 - \omega_c^2) = i\omega \frac{e}{m} E_x + \omega_c \frac{e}{m} E_y$$

Then, again using Ohm's law  $\vec{J} = \rho \vec{V} = \sigma \cdot \vec{E}$

$$\sigma_{xx} = \frac{i \frac{e}{m} \omega n_0}{\omega^2 - \omega_c^2} = \frac{\epsilon_0 \omega_p^2 \omega}{\omega^2 - \omega_c^2}$$

similarly for the other elements

$$\sigma_{xy} = \frac{\epsilon_0 \omega_c \omega_p^2}{\omega^2 - \omega_c^2}; \quad \sigma_{zz} = -i \frac{e}{m} \frac{\omega n_0}{\omega} = -\frac{i \omega \epsilon_0 \omega_p^2}{\omega^2}$$

$$\sigma_{xy} = -\sigma_{yx} ; \quad \sigma_{zx} = \sigma_{yz} \quad (\vec{\sigma} \text{ is antisymmetric})$$

So

$$\hat{\underline{\underline{\sigma}}} = \begin{bmatrix} 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} & i \frac{\omega_c}{\omega} \frac{\omega_p^2}{\omega^2 - \omega_c^2} & 0 \\ -i \frac{\omega_c}{\omega} \frac{\omega_p^2}{\omega^2 - \omega_c^2} & 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_p^2}{\omega^2} \end{bmatrix}$$

for more than one species  $\omega_p^2 \rightarrow \sum \omega_{ps}^2$   
 $\omega_c \rightarrow \omega_{cs}$

Note: to add collisions to force equation  
 $-c\omega \vec{V} + \chi_c \vec{V} = \frac{e}{m} (\vec{F} + \vec{V} \times \vec{B}_0)$   
 $-c(\omega + i\chi_c) \vec{V} = \dots$  But don't just substitute  $\omega + i\chi_c$  for  $\omega$  because some

$\omega$ 's come from force equation and some come from Maxwell's equations!

Now from wave equation

$$\underline{k} \times (\underline{k} \times \bar{E}) + \frac{\omega^2}{c^2} \underline{k} \cdot \bar{E} = 0$$

There are 2 basic solutions:

i)  $\underline{k} \parallel \bar{E}$  longitudinal waves: NOT possible  
in charge-free space

so  $\underline{k} \times \bar{E} = 0$  Then  $\underline{k} \cdot \bar{E} = 0$  gives  
the dispersion relation

and 2)  $\underline{k} \perp \bar{E}$  transverse waves

$$\text{then } |\underline{k} \times \bar{E}| = (|\underline{k}| \bar{E})$$

$$\underline{k} \times (\underline{k} \times \bar{E}) = (\underline{k} \cdot \bar{E}) \underline{k} - \underline{k}^2 \bar{E}$$

$$\text{so } \left( \underline{k}^2 \bar{E} - \frac{\omega^2}{c^2} \underline{k} \right) \bar{E} = 0$$

### EXAMPLES

1.  $\underline{k} \parallel \bar{E}$  and also take  $\underline{k} \parallel \underline{B}_0$

then  $\bar{E} \parallel \underline{B}_0$  and there is only an  $E_z$  component

result  $(1 - \frac{\omega_p^2}{\omega^2}) \frac{k_z^2}{k^2} = 0$  <sup>to wave</sup>

plasma oscillations (at  $T=0$ )

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2)  $E \parallel B_0$   $E + B_0$  assuming ions  
are only heavy (also see p. 129 Chen)

[For electrons only] we get 2 solutions

$$\text{L.H.P. (left hand polarized)} \quad k^2 = k_0^2 \left( 1 - \frac{\omega_p^2}{\omega^2} \frac{1}{1 + \frac{\omega_{ce}}{\omega}} \right)$$

$$\text{R.H.P. (right hand)} \quad k^2 = k_0^2 \left( 1 - \frac{\omega_p^2}{\omega^2} \frac{1}{1 - \frac{\omega_{ce}}{\omega}} \right)$$

$$\text{where } k_0^2 = \frac{\omega^2}{c^2} \quad \omega_{ce} = \left| \frac{eB}{m} \right|$$

[Read Chen Sectn 4.15 p. 126  
about cutoff's & Resonances]

Cutoff at  $k=0 \Rightarrow n = \frac{c}{v_0} = \frac{c}{\omega/k} \xrightarrow{\text{index of refraction}} \infty$

Resonance at  $k \rightarrow \infty \Rightarrow n \rightarrow \infty \quad n \propto \omega$

Waves are reflected at cutoff's and  
absorbed at resonances

for

L.H.P. solution cutoff at  $k=0$

$$\frac{\omega_p^2}{\omega^2} = 1 + \frac{\omega_{ce}}{\omega} \quad \text{but NO Resonance}$$

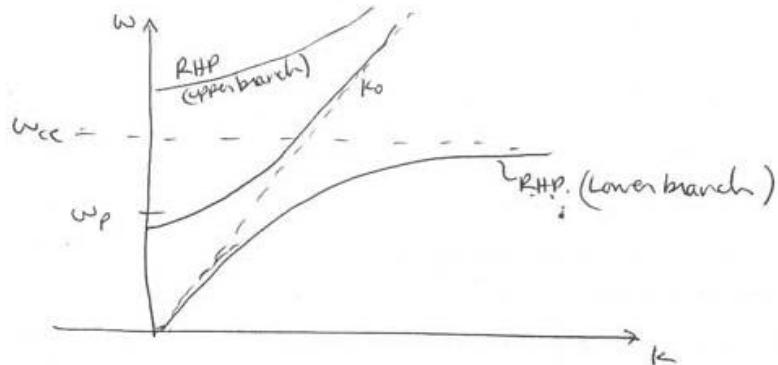
R.H.P.  $\downarrow$  over

$k \parallel B_0$      $E \perp B_0$     continued

R.H.P. solutn cutoff  $k=0$

$$\frac{\omega_p^2}{\omega^2} = 1 - \frac{\omega_{ce}}{\omega}$$

Resonance ( $k \rightarrow \infty$ ) at  $\omega = \omega_{ce}$



case for  $\omega_{ce} > \omega_p$

Now Add IONS (so we have electrons and ions)

L.H.P. solutn

$$k^2 = k_0^2 \left( 1 - \frac{\omega_{pe}^2}{\omega^2} \frac{1}{1 + \frac{\omega_{ci}}{\omega}} - \frac{\omega_{pi}^2}{\omega^2} \frac{1}{1 - \frac{\omega_{ci}}{\omega}} \right)$$

$$= k_0^2 \left( 1 - \frac{\omega_p^2 / \omega^2}{(1 + \frac{\omega_{ci}}{\omega})(1 - \frac{\omega_{ci}}{\omega})} \right) \quad \text{where } \omega_{ci} = \left| \frac{eB}{m_i} \right|$$

note:  $\omega_p^2 = \omega_{pe}^2 + \omega_{pi}^2$  where  $-\omega_{pe}^2 \frac{\omega_{ci}}{\omega} = +\omega_{pi}^2 \frac{\omega_{ce}}{\omega}$

$k \parallel B \quad E + B$  contd (example 2)

(electrons and ions continued)

we found

$$\text{LHP} \quad k^2 = k_0^2 \left( 1 - \frac{\omega_p^2/\omega^2}{(1 + \frac{\omega_{ce}}{\omega})(1 - \frac{\omega_{ci}}{\omega})} \right)$$

Similarly:

$$\text{RHP} \quad k^2 = k_0^2 \left( 1 - \frac{\omega_p^2/\omega^2}{(1 - \frac{\omega_{ce}}{\omega})(1 + \frac{\omega_{ci}}{\omega})} \right)$$

Examine cutoff's & Resonance:

	Cutoff	Resonance
LHP	$\frac{\omega_p^2}{\omega^2} = \left( 1 + \frac{\omega_{ce}}{\omega} \right) \left( 1 - \frac{\omega_{ci}}{\omega} \right)$	$\omega = \omega_{ci}$
RHP	$\frac{\omega_p^2}{\omega^2} = \left( 1 - \frac{\omega_{ce}}{\omega} \right) \left( 1 + \frac{\omega_{ci}}{\omega} \right)$	$\omega = \omega_{ce}$

For a VLF (very low frequency wave)

CASE I  
both LHP and RHP solutions

$$\left( 1 - \frac{\omega_c}{\omega} \right) \sim \frac{\omega_c}{\omega} \quad \text{so}$$

$$k^2 = k_0^2 \left( 1 + \frac{\omega_p^2}{\omega_c \omega_{ce}} \right)$$

Usually  $\omega_p^2 \gg \omega_{ce} \omega_{ci}$  so

$$\text{So } k^2 \approx k_0^2 \frac{\omega_p^2}{\omega_e \omega_{ci}}$$

$$V_\phi = \frac{\omega}{k} = C \left( \frac{\omega_e \omega_{ci}}{\omega_p^2} \right)^{1/2} \ll C$$

slow!

This wave has a phase velocity

$$V_\phi = \left( C^2 \frac{e^2 B^2}{m M} n_0 \right)^{1/2}$$

$$= \frac{B}{\sqrt{n_0 \rho}} = \text{Alfvén speed} \\ = V_A$$

wave travels along  $B$  field -

Not an EM wave just a magnetic wave

Case 2 Intermediate wave

$$\omega_{ci} \ll \omega \ll \omega_{ce}$$

$$\text{R.H.P. } k^2 = k_0^2 \left( 1 + \frac{\omega_p^2 / \omega^2}{\omega_e / \omega} \right)$$

$$= k_0^2 \left( 1 + \frac{\omega_p^2}{\omega \omega_e} \right)$$

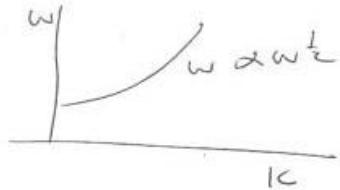
Let  $\omega_p^2 \gg \omega \omega_{ce}$  then  $k^2 \approx k_0^2 \frac{\omega_p^2}{\omega \omega_e}$

$$K = \frac{\omega}{c} n \quad n = \text{inductive refractive}$$

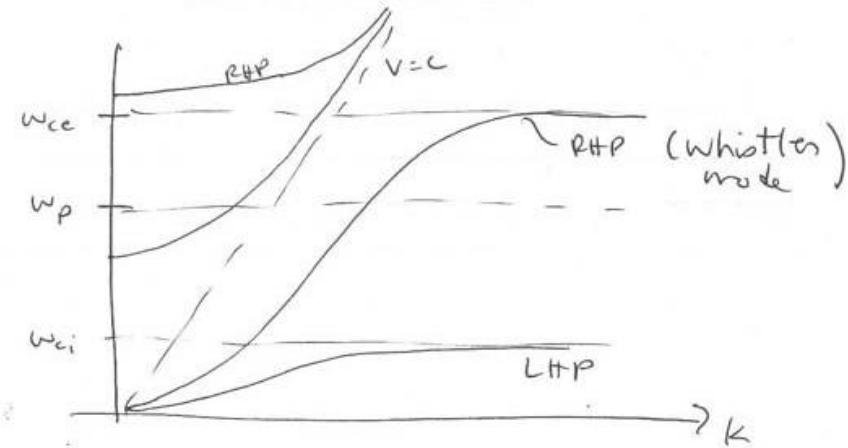
let  $\omega_p^2 \gg \omega \omega_{ce}$

$$V_\phi - \frac{\omega}{k} = c \left( \frac{\omega \omega_{ce}}{\omega_p^2} \right)^{1/2} \propto \omega^{1/2}$$

$$n = \sqrt{1 - \frac{\omega_p^2}{\omega \omega_{ce}}}$$



called a Whistler Wave



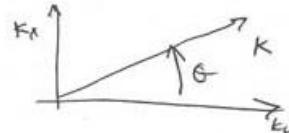
General wave equation

$$\vec{k} \times (\vec{k} \times \vec{E}) = \omega^2 \mu_0 \epsilon \cdot \vec{E} \quad \text{assume } \vec{B} = \hat{z} B_z$$

$$\vec{k} = k_x \hat{x} + k_z \hat{z}$$

$$\begin{bmatrix} k_z^2 & 0 & -k_x k_z \\ 0 & k_x^2 + k_z^2 & 0 \\ -k_x k_z & 0 & k_x^2 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \frac{\omega^2}{c^2} \begin{bmatrix} K_{\perp} & iK_x & 0 \\ -ik_x & K_{\perp} & 0 \\ 0 & 0 & K_{\parallel} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$$\text{now let } n^2 = \frac{k^2}{k_0^2}$$



$$\frac{k_z^2}{k_0^2} = n^2 \cos^2 \theta \quad \text{and} \quad \frac{k_x^2}{k_0^2} = n^2 \sin^2 \theta$$

for  $\begin{bmatrix} \quad \end{bmatrix}(\vec{E}) = 0$  The determinant must = 0

$$\det \begin{bmatrix} n^2 \cos^2 \theta - K_{\perp} & -ik_x & -n^2 \cos \theta \sin \theta \\ ik_x & n^2 - K_{\perp} & 0 \\ -n^2 \cos \theta \sin \theta & 0 & n^2 \sin^2 \theta - K_{\parallel} \end{bmatrix} = 0$$

Includes All waves in a cold plasma

for  $\vec{k} \parallel \vec{B}_0$  obtain RHP and LHP waves  
as we did above

$$\text{for } \vec{k} \perp \vec{B}_0 \quad \cos \theta = 0 \quad \sin \theta = 1$$

$$\begin{bmatrix} -k_+ & -ik_x & 0 \\ ik_y & n^2 - k_+ & 0 \\ 0 & 0 & n^2 - k_{\parallel} \end{bmatrix} = 0$$

where  $k_{\perp} = 1 - \frac{\omega_{pe}^2}{\omega^2 - \omega_{ci}^2} - \frac{\omega_{pi}^2}{\omega^2 - \omega_{ci}^2}$

$$k_x = \frac{\omega_{ce}}{\omega} \frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2} + \frac{\omega_{ci}}{\omega} \frac{\omega_{pi}^2}{\omega^2 - \omega_{ci}^2}$$
from  
Pichotri  
Ternon

$$(n^2 - k_{\parallel}) (k_+ (k_{\perp} - n^2) - k_x^2) = 0$$

2 solutions

$$1) n^2 - k_{\parallel} = 0 \quad k^2 = \frac{\omega^2}{c^2} \left(1 - \frac{\omega_{pe}^2}{\omega^2}\right) \text{ ORDINARY WAVE}$$

$$2) n^2 = \frac{k_{\perp}^2 - k_x^2}{k_+} \quad \text{extraordinary wave}$$

Cutoffs ( $n=0$ )

$$\omega = \omega_p \quad \text{Ordinary}$$

$$\frac{\omega_p^2}{\omega^2} = (1 \pm \frac{\omega_{ce}}{\omega})(1 \mp \frac{\omega_{ci}}{\omega}) \quad \text{extraordinary}$$

Resonance ( $n \rightarrow \infty$ )

$$\begin{aligned} \text{none} - \text{ordinary} \\ \text{extraordinary at } \frac{\omega_p^2}{\omega^2} \left(1 - \frac{\omega_{ce}\omega_{ci}}{\omega^2}\right) = \\ \left(1 - \frac{\omega_{ce}^2}{\omega^2}\right) \left(1 - \frac{\omega_{ci}^2}{\omega^2}\right) \end{aligned}$$

Look at extraordinary resonance in 2 limits

1)  $\omega^2 \gg \omega_{ce}, \omega_{ci}$       Upper hybrid Resonance

$$\frac{\omega_p^2}{\omega^2} = 1 - \frac{\omega_{ce}^2}{\omega^2} \quad \boxed{\omega^2 = \omega_p^2 + \omega_{ce}^2}$$

2)  $\omega_{ci}^2 \ll \omega^2 \ll \omega_{ce}^2$

$$\frac{\omega_p^2}{\omega^2} = \omega_p^2 \frac{\omega_{ce}\omega_{ci}}{\omega^2} = \frac{\omega_{ce}^2}{\omega^2} \quad \text{Lower hybrid Resonance (LHR)}$$

$$\omega^2 = \frac{\omega_p^2 \omega_{ce} \omega_{ci}}{\omega_p^2 + \omega_{ce}^2} = \begin{cases} \omega_{ce}\omega_{ci} & \text{for } \omega_{ce}^2 \ll \omega_p^2 \\ \omega_p^2 & \text{for } \omega_{ce}^2 \gg \omega_p^2 \end{cases}$$