7. More Waves

7.1. Phase and group velocities in higher dimensions

In this chapter, we will study waves that have more complicated phase and group velocity behavior than the surface waves discussed in the last chapter. We therefore need to generalize phase and group velocity to waves that propagate in two and three dimensions. We begin by generalizing the concept of wavenumber k to the wave vector $\mathbf{k} \equiv k\hat{\mathbf{x}} + l\hat{\mathbf{y}} + m\,\hat{\mathbf{z}}$. A three-dimensional plane wave can then be expressed as

$$q = A\cos(kx + ly + mz - \sigma t) = A\cos(\mathbf{k}\cdot\mathbf{x} - \sigma t)$$

Here q is some quantity such as pressure or a displacement or velocity component and $\mathbf{x}$ is the position. The phase velocity is again the velocity at which an observer must move to keep the phase, $\phi = \mathbf{k}\cdot\mathbf{x} - \sigma t$, constant. This obviously has the same direction as $\mathbf{k}$ (see Figure 42). Thus we can write

$$\mathbf{c}_p = |c_p|\hat{\mathbf{n}}_k$$

where

$$\hat{\mathbf{n}}_k = \frac{\mathbf{k}}{|\mathbf{k}|}$$

is unit vector in the direction of $\mathbf{k}$, and

$$|c_p| = \frac{\sigma}{|\mathbf{k}|}$$

$$|\mathbf{k}| = \sqrt{k^2 + l^2 + m^2}$$

In a similar manner, the vector group velocity becomes the gradient of the dispersion relation in ‘‘k-space’’

$$\mathbf{c}_g = \frac{\partial\sigma}{\partial k}\hat{\mathbf{x}} + \frac{\partial\sigma}{\partial l}\hat{\mathbf{y}} + \frac{\partial\sigma}{\partial m}\hat{\mathbf{z}} \equiv \nabla_k\sigma$$

which does not have to be in the same direction as $\mathbf{k}$ or $\mathbf{c}_p$. The two-dimensional analogue of these expressions is recovered by simply setting the terms associated with the un-wanted component to zero.

7.2. Sound waves

We now briefly consider waves in a compressible fluid. If we ignore gravity, rotation and viscosity and let density be variable, we can let

$$\rho = \rho_0 + \varepsilon\rho_1 + \varepsilon^2\rho_2 + \varepsilon^3\rho_3 + \cdots$$

The first order momentum equation for a perturbation about $\mathbf{u}_0 = 0$ becomes

$$\rho_0 \frac{\partial u_1}{\partial t} + \nabla p_1 = 0$$

Taking the divergence of this equation, ignoring second order terms, gives

$$\rho_0 \frac{\partial}{\partial t} (\nabla \cdot \mathbf{u}_1) + \nabla^2 p_1 = 0$$

The first order conservation of mass equation (see section 2.2.3) becomes

$$\frac{\partial \rho_1}{\partial t} + \rho_0 (\nabla \cdot u_1) = 0$$

Using the definition of compressibility, β (see section 3.2), and assuming that it is constant, we can write

$$\beta \rho_0 \frac{\partial p_1}{\partial t} = \frac{\partial \rho_1}{\partial t}$$

(to first order). Thus the mass conservation equation becomes

$$\beta \frac{\partial p_1}{\partial t} + \nabla \cdot u_1 = 0$$

which can be used to eliminate $\nabla \cdot \mathbf{u}_1$ in the momentum equation. We get

$$\beta \rho_0 \frac{\partial^2 p_1}{\partial t^2} - \nabla^2 p_1 = 0$$

Finally, substituting a plane wave, $p_1 = P \cos (kx - \sigma t)$, we obtain the dispersion relation

$$\sigma = \frac{k}{\sqrt{\rho_0 \beta}}$$

The phase velocity for these sound waves is

$$c_p = \frac{1}{\sqrt{\rho_0 \beta}}$$

which proves a result used in section 3.2.

Like shallow water surface waves, sound waves are non-dispersive. The temperature, salinity and pressure-dependence of sound speed in the ocean conspire to create a minimum in sound speed at a depth of about 1 km (see Figure 43(a)). This low velocity zone traps sound energy generated within it and permits the sound energy to propagate vast distances (because the amplitude falls off as range squared rather than cubed). There are many possible ray paths between any source and receiver pair In a flat ocean, the slowest ray would travel exactly along the axis of minimum sound speed and is called the axial ray. Other rays have paths which oscillate up and down with rays that have their extremal points furthest from the sound minimum having the least travel time. There would furthermore be an infinite number of rays, with most of them having angles with respect to the axis, which are small. Due to the curvature of the earth, however, no ray can continuously follow the sound minimum. There will therefore only be a finite number of rays for a given sour-receiver configuration and the axial ray is defined as the ray that remains closer to the sound minimum then any other ray (see Figure 43(b)). The fact that different sound rays sample different depths makes it possible to determine the average vertical profile of sound speed in a vertical slice between a source and receiver. This is called ocean acoustic tomography (OAT). If, additionally, travel time measurements are made for both directions of travel, one can use the Doppler-shift to simultaneously determine the average fluid motion in the vertical slice. Using arrays of sources and receivers (in both depth and horizontal position) OAT has the potential to determine the three-dimensional oceanic structure much more rapidly than can a ship. OAT also provides spatial averages of the structure and is thus a more appropriate (and perhaps more cost-effective) way to study larger scale phenomena (such as eddies) than is an array of current meters. The large scale averaging potential of OAT has particular relevance to the problem of measuring the slow warming of the ocean that should be expected if our planet is warming up due to the greenhouse effect.

7.3. Rossby waves

As I have stated, there are many similarities between all waves. Once you have identified the restoring force, it is usually a simple matter to equate it to the fluid acceleration and deduce first the dispersion relation and then the phase and group velocities. However, not all waves have the simple behavior observed for surface (and other waves like sound) in which the phase and group velocities are in the same direction. An example is the wave associated with deviations from geostrophic balance.

The fundamental premise of geostrophic balance is that horizontal pressure gradients are balanced by the horizontal component of the Coriolis force. We already know that this requires large horizontal scales and we have previously argued that a consequence of this balance is a strong tendency for the fluid to move as ‘‘vortex tubes’’ that always remain parallel to the rotation axis. Consider a meander in an eastward geostrophic current in the Northern Hemisphere (see Figure 44(a)). Because the Earth is curved, fluid displaced northwards requires shortened vortex tubes, while fluid that is displaced south requires stretched tubes (see Figure 44(b)). The conservation of angular momentum for a northward displaced tube requires that its rotation must slow down and it will find itself rotating clockwise relative to its environment (see Figure 44(c)). This clockwise rotation will induce northward flow of fluid to the west and southward flow of fluid to the east. The conservation of angular momentum for these induced flows results in secondary flows which will sweep the original displaced vortex tube back to its original position (again, see Figure 44(c)). The fact that the initial northward displacement propagates westward implies that the phase velocity of the wave associated with this restoring mechanism is to the west. A tube which is initially displaced south will also be restored to its original position and will also imply a westward phase velocity.

It is not easy to estimate the magnitude of the restoring force directly from the above physical argument primarily because the flows are fundamentally two-dimensional. However, it is fairly straightforward to derive the desired result directly from the momentum equation for geostrophic balance modified to allow time variations

$$\frac{D\mathbf{u}}{Dt} + 2\Omega \times \mathbf{u} = \frac{1}{\rho} \nabla p$$

Consider a small deviation $\mathbf{u}'$ to a uniform steady eastward geostrophic current $\mathbf{u}_0 = U_0\hat{\mathbf{x}}$. If we assume that vertical motions are essentially zero, we can let $\mathbf{u} = (u' + U_0)\hat{\mathbf{x}} + v'\hat{\mathbf{y}}$ with $\hat{\mathbf{x}}$ east and $\hat{\mathbf{y}}$ north (hence $\hat{\mathbf{z}}$ upwards) and we can write

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}'}{\partial t} + U_0 \frac{\partial}{\partial x} \mathbf{u}'$$

The components of the modified geostrophic balance equation can then be written

$$\left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x}\right)u' - f v' = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x}\right)v' + f(u' + U_0) = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

where the Coriolis parameter $f \equiv 2\Omega\cos(\theta)$. Now assume that the unperturbed current satisfies geostrophic balance

$$0 = -\frac{1}{\rho}\frac{\partial P_0}{\partial x}$$

$$fU_0 = -\frac{1}{\rho}\frac{\partial P_0}{\partial y}$$

If we subtract these equations from the previous two and define the pressure perturbation $p' = p - P_0$, we obtain

$$\left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x}\right)u' - f v' = -\frac{1}{\rho}\frac{\partial p'}{\partial x}$$

$$(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x})v' + f u' = -\frac{1}{\rho}\frac{\partial p'}{\partial y}$$

The pressure deviation can be eliminated from the problem by taking the y partial derivative of the first equation and subtracting it from the x partial derivative of the second equation. The result is

$$(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x})(\frac{\partial u'}{\partial y} - \frac{\partial v'}{\partial x}) - \beta\, v' = 0$$

In writing this we have defined $\beta \equiv \frac{\partial f}{\partial y}$ and used the fact that if the vertical velocity is zero and the fluid is incompressible, north-south stretching must compensate for east-west squeezing, i.e.

$$\frac{\partial u'}{\partial x} = -\frac{\partial v'}{\partial y}$$

The term $\frac{\partial u'}{\partial y} - \frac{\partial v'}{\partial x}$ is the component of vorticity perpendicular to the Earth's surface $(\nabla \times \mathbf{u})$. The term $\beta\, v$ is the rate of change of vorticity due to the squeezing (stretching) of a vortex tube as it moves north (south) in the Northern Hemisphere of a curved Earth. Thus the above equation is simply a statement of the conservation of angular momentum on a curved earth.

The value of β can be easily derived. Suppose θ increases (southward displacement) by an angle α (see Figure 44(b)).

$$f(\theta + \alpha) = 2\Omega \cos(\theta + \alpha) = 2\Omega[cos(\theta)\cos(\alpha) - \sin(\theta)\sin(\alpha)]$$

where I have used a standard trigonometric identity. For small α, $\cos(\alpha) \approx 1$ and $\sin(\alpha) \approx \alpha \approx -\frac{y}{\alpha}$, where -y is the southward displacement and a is the radius of the Earth. Thus

$$f(\theta + \alpha) = 2\Omega \cos(\theta) + \frac{2\Omega \sin(\theta)}{a}\, y = f(\theta) + \beta y$$

and obviously $\frac{\partial f}{\partial y} = \beta$ with $\beta = \frac{2\Omega \sin(\theta)}{a}$.

Taking the x derivative of the vorticity conservation equation and the y derivative of the mass conservation equation, we can easily eliminate u' and obtain

$$(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x})[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}]\, v' + \beta \frac{\partial v'}{\partial x} = 0$$

Substituting a ‘‘wave’’ $v' = V \sin(kx - \eta t)$ whose crests and troughs run north-south (corresponding to a sinusoidal meander in the original eastward current) into the above equation gives

$$(\eta - U_0 k) k^2 V \cos(kx - \eta t) = -\beta k \cos(kx - \eta t)$$

The dispersion relation is

$$\eta = U_0 k - \frac{\beta}{k}$$

and

$$c_p = \frac{\eta}{k} = U_0 - \frac{\beta}{k^2} < U_0$$

$$c_g = \frac{\partial \eta}{\partial k} = U_0 + \frac{\beta}{k^2} > U_0$$

Note that the effect of the background velocity U_0 is simply to sweep the disturbance along. We could, in fact, have simplified the algebra if we had shifted into a coordinate system moving along with the basic flow. IN this moving coordinate system, the magnitude of the phase and group velocities is the same, but c_p is always *westward* (because $\beta \geq 0$) while c_g is always *eastward*. At $\theta = 45^o$, $\beta = 1.6 \times 10^{-11}\ (m\ sec)^{-1}$. With a wavelength of 300 km ($k = 2.1 \times 10^{-5}\ m^{-1}$) the wave has a period $\frac{2\pi}{\eta} = 90$ days and a phase velocity $c_p = 4$ cm/s. It is extremely slow moving relative to the speed of the unperturbed current (typically 10-300 cm/s).

These waves are called Rossby or Planetary waves. Because of their very large scale, their very slow phase and group velocities and the effects of stratification and finite amplitudes, which result in non-linear behavior, they are difficult to find in the form considered in our discussion. The spectacular waves seen in the atmosphere of Jupiter are closely related to Rossby waves and they play an important role in weather systems on Earth. However, the meanders in the surface currents such as the Gulf Stream after it has separated from the coast north of Cape Hatteras (Figure 45) are probably the closest situation to our analysis. The strong stratification at the thermocline inhibits vertical motion sufficiently that our physical argument based on the conservation of angular momentum is still approximately correct. The eastward component of the current substantially exceeds the westward phase velocity, so that the meanders (see Figure 46) are swept along by the current and appear to move physically eastward. The westward phase velocity can be deduced from the fact that the current flows through the meanders and thus the phase of the meanders must be traveling eastward at a slower rate than the current.

Meanders that grow so large that they pinch off to form rings can maintain their existence for a long time and drift into areas of the ocean which are much less energetic (particularly to the south). In this situation, it has been possible to actually see the westward phase and eastward group velocity. The problem of course has been to sufficiently instrument a patch of ocean to observe the motion of a geostrophic eddy (i.e. Rossby wave group) over time.

7.4. Stably stratified fluids

7.4.1. Internal interface waves

When a perturbation occurs at an internal interface between two fluids of constant density $\rho_1 > \rho_2$ (ρ_2 on top), the restoring pressure becomes

$$p = (\rho_1 - \rho_2)gh$$

This expression differs from that at the top surface only because ρg has been replaced by the much smaller $(\rho_1 - \rho_2)g$. We can thus have waves on the interface that differ from surface waves primarily in having a lower frequency for a given wavelength. In fact, if there is only one internal interface and the layers above and below the interface are thick compared to $\frac{\lambda}{2}$, the interface wave solution is identical to that for deep water surface waves, aside from the reduced effect of gravity.

Interface waves are commonly seen in fjords or estuaries where fresh water overlies salt water and they cause a phenomenon called dead water. A moving ship will generate interface as well as surface waves. The energy required to generate the interface waves can be substantial and the ship will have difficulty progressing with no reason apparent at the surface. Once the ship's speed exceeds the phase velocity of the fastest interface waves (which will be much slower than the surface waves), it leaves them behind and the extra drag suddenly disappears.

Things are more complicated if either of the layer thickness is comparable to the wavelength, but the mathematics of finding the appropriate solution is very similar to that for surface waves on a fluid of finite depth. However, if there are two interfaces, there is the possibility of different modes: one in which both interfaces oscillate up and down together and one in which the upper interface is going down at places where the lower interface is going up. We will not consider the details of such solutions, but instead, study the more interesting case of internal waves in density which varies continuously with depth.

7.4.2. Brunt-Vaisala frequency

Suppose that a parcel of fluid of density ρ_0 is displaced upwards by a distance $\xi = z - z_0$ in a fluid whose density $\rho = \rho(z)$ (see Figure 47(a)). The restoring force (per unit volume) on the parcel will be $-\Delta\rho g\xi\,\hat{\mathbf{z}}$, where $\Delta\rho$ is the density difference between the displaced particle and the new environment in which it finds itself. As long as we cross no interface where ρ is discontinuous, we can expand ρ in a Taylor series about $z = z_0$ and write

$$\rho(\xi) = \rho_0 + \left[\frac{d\rho}{dz} \right]_0 \xi + higher\ order\ terms$$

where the subscript means evaluated at $z = z_0$. If ξ is small, we have

$$\Delta\rho = \rho_0 \left[\frac{d\rho}{dz} \right]_0 \xi$$

Equating the density times the vertical acceleration on the displaced parcel to the restoring force, we finally have

$$\rho \frac{d^2\xi}{dt^2} = -\, g \frac{d\rho}{dz}\ \xi$$

where I have dropped the subscript 0. You should recognize this as the equation for simple harmonic motion with solution $\xi = Ae^{iNt}$, where

$$N = \pm \sqrt{-\frac{g}{\rho}\ \frac{d\rho}{dz}}$$

is called the Brunt-Vaisala frequency. Note that if $\frac{d\rho}{dz} > 0$ (density increasing upwards), N is imaginary and the solution is no longer oscillatory, but instead is growing or decaying exponentials.

7.4.3. Internal waves

In the more general case where the displacement is at an angle θ with respect to the horizontal, the vertical displacement is is $\xi \sin(\theta)$ and the component of the restoring force along the direction of displacement is $\xi \sin^2(\theta) \frac{d\rho}{dz}$ (see Figure 47(b)). Thus

$$\frac{d^2\xi}{dt^2} = N^2 \sin^2(\theta)\xi$$

We immediately see that the dispersion relation for these internal waves must be

$$\eta = \pm\, N|\sin(\theta)|$$

The absolute value symbol around $sin(\theta)$ simply recognizes that it does not matter whether the plane of displacement is tilted up or down with respect to the horizontal. We can conclude immediately that the frequency of an internal wave can vary from zero (pure horizontal motion) to the Brunt-Vaisala frequency (pure vertical motion). The dispersion relation can be interpreted in two ways. If you excite the fluid at a certain frequency between 0 and N, the motion of the fluid in the wave will have a specific angle with respect to the horizontal. On the other hand, if you have or want motion at a specific angle, you must excite it at a specific frequency.

Because internal waves involve motion tilted with respect to the horizontal, the wave oscillation will be in the vertical as well as horizontal direction, and we need to use the two-dimensional generalization of the wave number to a wave vector. For motion in the x-z plane, $\mathbf{k} = k\hat{\mathbf{x}} + m\hat{\mathbf{z}}$. The fluid velocity components for a plane wave with crests perpendicular to $\mathbf{k}$ are

$$w = \mathrm{Re}[A\, e^{i(kx + mz - i\eta t)}]$$

$$u = Re[B\, e^{i(kx + mz - i\eta t)}]$$

Substituting these into the mass conservation equation

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad \rightarrow \quad iku + imw = 0 \quad \rightarrow \quad \mathbf{k} \cdot \mathbf{u} = 0$$

Thus the wave vector must be perpendicular to the fluid velocity (see Figure 48(a)). Using the dispersion relation and the first section of this chapter, one can easily conclude that

$$\mathbf{c}_p = \pm \frac{N|\sin(\theta)|}{k}\,\hat{\mathbf{k}} = \pm \frac{N|\sin(\theta)|}{|\mathbf{k}|^2}\,\mathbf{k}$$

Thus the phase velocity will also be perpendicular to the fluid velocity.

The generalization of group velocity is

$$\mathbf{c}_g = \nabla_k \eta$$

Since η depends only on θ it is convenient to write the gradient in k-space in polar coordinates

$$\nabla_k = \hat{\mathbf{k}}\,\frac{\partial}{\partial|\mathbf{k}|} + \frac{\hat{\theta}}{|\mathbf{k}|}\,\frac{\partial}{\partial\theta}$$

where $\hat{\mathbf{k}} = \hat{x}\sin(\theta) + \hat{z}\cos(\theta)$ is the unit vector parallel to $\mathbf{k}$ and $\hat{\theta} = \hat{x}\cos(\theta) - \hat{z}\sin(\theta)$ is the unit vector perpendicular to $\mathbf{k}$ The first term in the k-space gradient will be zero and we immediately have

$$\mathbf{c}_g = \pm \frac{N}{|\mathbf{k}|}\,|\cos(\theta)|\,\hat{\theta}$$

which is perpendicular to **k** and thus parallel to **u**. This rather surprising result means that a group of these waves has the peculiar behavior that phase fronts (crests and troughs) propagate sideways across a patch of wave energy moving along as a group (see Figures 48(a), 49(a)). We also see that $\cos(\theta)$ goes to zero as the motion becomes purely vertical and the frequency approaches the Brunt-Vaisala frequency. Thus purely vertical motion does not result in the propagation of energy. Figure 49(b) demonstrates that no wave is generated when $\eta > N$.

Internal waves can be seen in the atmosphere when they are made visible by the condensation in rising air (wave or lenticular (shaped like lenses) clouds). They are often generated by obstacles such as mountains (the one behind Mt. Rainier typically is shaped like a boomerang) and they are then referred to as lee waves, but they can also be generated by air flowing over cumulus clouds or weather fronts and by instabilities associated with vertical shear of the wind (Kelvin-Helmholtz instability). Internal waves also occur in the ocean. Like surface waves, the internal wave solution is barely affected by viscosity, so that internal waves can propagate over long distances. For this reason and because it is difficult to look into the ocean and actually observe internal waves, little is known about the precise mechanisms for generating most of the internal wave energy in the ocean.

7.4.4. Boussinesq approximation

A more rigorous derivation of the results just obtained comes from substituting the two-dimensional plane wave into the full system of equations for a stratified fluid. In doing this, we resort to a very important simplification called the Boussinesq approximation. In this approximation, the fluid is assumed to have constant density *except* where the density perturbation is multiplied by the large acceleration of gravity. This approximation can be rigorously justified in a wide variety of geophysically important situations by means of a perturbation expansion.

We need to modify the perturbation expansion of Chapter 6 to allow for variable density. In our discussion of non-dimensionalization in Chapter 5, we have already considered what happens to the momentum equation when there is a small perturbation ρ_1 to the basic density ρ_0. We shall also assume that $\mathbf{u}_1$ is a small perturbation from the state of rest (i.e. $\mathbf{u}_0 = 0$) and thus p_1 is the deviation from hydrostatic pressure. We shall additionally ignore the effects of rotation and viscosity (i.e. assume that the internal wave period is small compared to a day or to the diffusion time for disturbances whose scale is the wavelength). These last two assumptions are not a necessary part of the Boussinesq approximation. The first order Navier-Stokes equation becomes

$$\frac{\partial \mathbf{u}_1}{\partial t} + \frac{1}{\rho_0} \nabla p_1 = \sigma_1 \hat{\mathbf{z}}$$

(see section 5.1), where I have used the buoyancy perturbation defined by

$$\sigma_1 \equiv \frac{\rho_1}{\rho_0} g$$

Clearly we can recover ρ_1 once σ_1 is known, so we do not have to specifically worry about ρ_0 in the buoyancy term. Thus the $\rho_0(z)$ multiplying the pressure gradient is the only remaining barrier to ignoring the variation of density except in the buoyancy term. Note, however, that since $\mathbf{u}_0 = 0$, we must have $\rho_0 = \rho_0(z)$ and therefore

$$\nabla \frac{p_1}{\rho_0} = \frac{1}{\rho_0} \nabla p_1 + p_1 \nabla\left(\frac{1}{\rho_0}\right) = \frac{1}{\rho_0} \nabla p_1 - \frac{p_1}{\rho_0{}^2} \frac{d\rho_0}{dz} \hat{\mathbf{z}}$$

Thus, if the vertical amplitude A of the motion is small relative to the scale height H for significant variation in the basic density profile,

$$\frac{\partial p_1}{\partial z} \approx \frac{p_1}{A} \gg \frac{p_1}{H} \approx \frac{p_1}{\rho_0} \frac{d\rho_0}{dz}$$

and hence we can ignore the last term on the right side above in comparison to the $\hat{\mathbf{z}}$ component of the first term on the right. Thus

$$\frac{1}{\rho_0} \nabla p_1 = \nabla \frac{p_1}{\rho_0}$$

Consequently, for motions with small vertical amplitudes, we can take ρ_0 inside the gradient operator in the pressure term of the momentum equation just as if it were a constant. Once this is done, we can treat $\frac{p_1}{\rho_0}$ as a new variable from which p_1 can be recovered after we have solved for $\frac{p_1}{\rho_0}$.

The density of ocean water differs from that of fresh water by only about 2%. Thus the maximum density change over the full depth will be less than this even in an estuary. Thus H is always much larger than the water depth and the necessary condition that $A \ll H$ will always be satisfied. The main vertical density variations in the atmosphere are due to compressibility. Since the scale height is about 10 km (see section 3.2) and internal wave amplitudes are typically less than 1 km, the approximation should also be reasonably good under most circumstances in the atmosphere.

Turning now to the conservation of mass equation. We first note that spatially variable density is completely consistent with an assumption of incompressibility. Thus we can still use

$$\nabla \cdot \mathbf{u} = 0$$

if the fluid is incompressible.

Of course real fluids are never incompressible. Using the definition of compressibility β (section 3.2) and referring to the discussion on sound waves in section 6.2.5,

$$\frac{Dp}{Dt} = c^2 \frac{D\rho}{Dt}$$

where c is the speed of sound. If we non-dimensionalize using $p = \rho_0 U^2 p'$, $t = \tau t'$ and $\rho = \rho_0 \rho'$, this equation becomes

$$\frac{U^2}{c^2}\frac{Dp'}{Dt'} = \frac{D\rho'}{Dt'} \to 0 \;\; when \;\; (\frac{U}{c})^2 \to 0$$

Thus density fluctuations caused by the dynamic pressure (i.e. the kinetic energy) of the flow will be negligible if the $(Mach\ number)^2$ is small. The Mach number is always small in the ocean (except near an explosion) and only rises to 0.1 in the atmosphere in some Hurricanes and in the Jet Stream. Even then, dynamic compressibility effects are small because the Mach number enters as its square and it never exceeds 0.3. The only time that we need to worry about the dynamic effects of compressibility in natural atmospheric phenomena is during volcanic eruptions.

The two components of the momentum equation plus the continuity equation give us three equations for the *four* unknowns, u, w, $\frac{p_1}{\rho_0}$, and σ_1. The fourth can be derived by noting that incompressibility also implies that

$$\frac{D\rho}{Dt} = \frac{D}{Dt}(\rho_0 + \rho_1) = \frac{D\rho_0}{Dt} + \frac{D\rho_1}{Dt} = w_1\frac{d\rho_0}{dz} + \frac{\partial\rho_1}{\partial t} = 0$$

where w_1 is the first order vertical velocity. In writing this equation, I have discarded second order terms and used the fact that $\rho_0 = \rho_0(z)$ and hence $\frac{\partial\rho_0}{\partial t} = 0$. Explicit in this perturbation expansion is the assumption that $\rho_1 \ll \rho_0$, which again requires the vertical amplitude of the motion to be small compared to the scale height for significant variations of the basic density profile. (Actually it requires that the scale height for *potential density* variations be large compared to the vertical amplitude.)

Multiplying the last equation by $\frac{g}{\rho_0}$ gives

$$\frac{\partial\sigma_1}{\partial t} + N^2 w_1 = 0$$

Where N is the Brunt-Vaisala frequency for the the zeroth order density $\rho_0(z)$.

To re-capitulate: the conditions required for the Boussinesq approximation are

(1) The fluid velocity is small compared to the speed of sound.

(2) The vertical amplitude of the motion is small compared to the scale height of the variations of the unperturbed density.

Dropping the subscript 1, the complete set of of first order equations under the Boussinesq approximation in a non-rotating, inviscid fluid are

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{p}{\rho_0}\right) = 0$$

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial z}\left(\frac{p}{\rho_0}\right) - \sigma = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial \sigma}{\partial t} + N^2 w = 0$$

7.4.5. More internal waves

At this point there are two ways to proceed. We can cross-differentiate the horizontal and vertical components of the momentum equation and subtract (equivalent to taking the curl of the vector momentum equation) to eliminate $\frac{p}{\rho_0}$. The result is an equation for the component of the vorticity perpendicular to the x-z plane

$$\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \frac{\partial \sigma}{\partial x} = 0$$

Note that vorticity can be generated in a density variable fluid by the horizontal gradient of the buoyancy as well as by viscosity. This is physically obvious because a gravity field acting on horizontally adjacent parcels of fluid with different density imparts a twisting torque to the fluid (see Figure 50). We can take the time derivative of this vorticity equation and use the fourth equation above to eliminate σ. Finally, if we assume that N is constant, we can use appropriate x and z derivatives and the continuity equation to eliminate u giving

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)w + N^2 \frac{\partial^2 w}{\partial z^2} = 0$$

We can then substitute a plane wave $w = W \cos(kx + mz - \eta t)$ into this equation to find

$$\eta^2(k^2 + m^{2)} = N^2 k^2$$

and thus the dispersion relation

$$\eta = \pm \frac{Nk}{\sqrt{(k^2 + m^2}} = \pm N \frac{k}{|k|} = \pm N|\sin(\theta)|$$

(see Figure 48(b)), which is the same result derived earlier. Calculation of $\mathbf{c}_p$ and $\mathbf{c}_g$ can proceed as before, although one can also, at the cost of tedious algebra, do the k-space derivatives in the rectangular (k,m) coordinates.

An alternate way to get to the same results is to substitute a plane wave into the system of equations prior to reducing them to a single equation. When you do this,

$$\frac{\partial}{\partial x} \rightarrow ik \qquad \frac{\partial}{\partial z} \rightarrow im \qquad \frac{\partial}{\partial t} \rightarrow -i\eta$$

and you are left with as system of algebraic equations that can be solved for η.