NOTES AND CORRESPONDENCE

Notes on a Catastrophe: A Feedback Analysis of Snowball Earth

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ABSTRACT

The language of feedbacks is ubiquitous in contemporary earth sciences, and the framework of feedback analysis is a powerful tool for diagnosing the relative strengths of the myriad mutual interactions that occur in complex dynamical systems. The ice albedo feedback is widely taught as the classic example of a climate feedback. Moreover, its potential to initiate a collapse to a completely glaciated snowball earth is widely taught as the classic example of a climate ''tipping point.'' A feedback analysis of the snowball earth phenomenon in simple, zonal mean energy balance models clearly reveals the physics of the snowball instability and its dependence on climate parameters. The analysis can also be used to illustrate some fundamental properties of climate feedbacks: how feedback strength changes as a function of mean climate state; how small changes in individual feedbacks can cause large changes in the system sensitivity; and last, how the strength and even the sign of the feedback is dependent on the climate variable in question.

1. Introduction

Early efforts to represent the earth's climate with energy balance models (EBMs) uncovered the disconcerting possibility that a relatively small decrease in the solar output might lead to a catastrophic global glaciation—the result of a runaway ice albedo feedback (e.g., Budyko 1969; North 1975; Lindzen and Farrell 1977). Although the issue remains controversial (e.g., Kerr 2000; Fairchild and Kennedy 2007; Allen and Etienne 2008), assorted lines of geological evidence appear to indicate that earth passed through several episodes of complete, or near-complete, glaciation during the Proterozoic (e.g., Kirschvink 1992; Hoffman et al. 1998; Hoffman and Li 2008). Follow-up integrations of more-comprehensive global climate models have also found climate states with a global or near-global glaciation, though they typically require larger reductions in the solar output than the earlier calculations suggested (e.g., Crowley and Baum 1993; Jenkins and Smith 1999; Crowley et al. 2001; Voigt and Marotzke 2009).

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To our knowledge, the factors controlling snowball earth have never been presented in terms of a formal feedback analysis and doing so provides an opportunity to demonstrate several basic properties of feedbacks. Applying this analysis to the original zonal mean energy balance climate models, the physical mechanism of the runaway glaciation can be clearly and simply demonstrated. The strength of the feedback is shown to equal the ratio of competing stabilizing and destabilizing tendencies on the global energy balance or, equivalently, competing tendencies on the local energy budget at the advancing ice line. The phenomenon of a snowball earth is a simple illustration of how climate sensitivity and feedback strength can change as a function of the mean climate state, which is an issue of some relevance for future climate predictions. Moreover, although there are obvious caveats because of the simplifying assumptions of the models, the instability is also an interesting example of a climate ''tipping point.''

The analytical solutions for the simple energy balance models permit feedback strengths to be calculated even for the unstable equilibrium climates. Doing so gives the somewhat counterintuitive but explainable result that the ice albedo can—under some conditions—behave as a negative feedback on global mean temperature. The cause is the peculiar physics of the small ice cap instability (e.g., North 1975), and that of a previously unreported counterpart at low latitudes.

2. Analysis

We begin with the classic equation for the annual mean, zonal mean EBM as a function of latitude (e.g., Budyko 1969; North 1975; Lindzen 1990):

$$\frac{Q}{4}S(x)[1-\alpha(x)] = A + BT(x) + \nabla\cdot\mathbf{F}, \qquad (1)$$

where Q is the solar constant and x is sine of latitude; $T(x)$, $S(x)$, and $\alpha(x)$ are the local temperature, the normalized latitudinal distribution of insolation, and the albedo, respectively; $A + BT(x)$ is a linearization of the outgoing longwave radiation (OLR); and $\mathbf{F}$ is the poleward heat transport.

Equation (1) can be integrated from equator to pole to give an expression for the global energy balance,

$$\frac{Q}{4}(1-\alpha_p) = A + B\overline{T}, \qquad (2)$$

where the overbar denotes the global mean, and α_p is the global average albedo:

$$\alpha_p \equiv \int_0^1 \alpha(x)S(x)\,dx. \qquad (3)$$

Last, let x_s be the latitude of the ice line (i.e., where $T = T_s$).

To a good approximation, $S(x)$ may be represented as $S(x) = 1 + s_2P_2(x)$, where $s_2 = -0.482$ and P_2 is the second Legendre polynomial: $P_2 = 1/2(3x^2 - 1)$ (e.g., Chylek and Coakley (1975); Fig. 1a). We adopt parameter values from Lindzen and Farrell (1977): $A = 211.1$ W m^{-2}; $B = 1.55$ W m^{-2} °C^{-1}. Note that the unit of T is degrees Celsius. We allow Q to vary in the vicinity of the modern-day value, which Lindzen and Farrell took to be $Q_0 = 1336$ W m^{-2}.

If α_p = constant, then x_s and $\overline{T}$ respond directly (with no feedbacks) to variations in Q. A feedback can be introduced by allowing albedo to be a function of temperature: an ice-free albedo α_1 is assumed for temperatures greater than T_s (typically -10°C) and an ice-covered albedo α_2 is assumed for temperatures less than T_s. Following Lindzen and Farrell (1977), we take $\alpha_1 = 0.3$ and $\alpha_2 = 0.6$. Therefore, from Eq. (3),

$$\alpha_p(\overline{T}) = \alpha_p[x_s(\overline{T})] = \alpha_1\int_0^{x_s} S(x)\,dx + \alpha_2\int_{x_s}^1 S(x)\,dx. \qquad (4)$$

Using the relationship between Legendre polynomials that $(2n + 1)P_n(x) = d/dx[P_{n+1}(x) - P_{n-1}(x)]$ (e.g., Abramowitz and Stegun 1965), Eq. (4) can be written as

$$\alpha_p(x_s) = \alpha_2 + (\alpha_1 - \alpha_2)\left[x_s + \frac{s_2}{5}(P_3(x_s) - P_1(x_s))\right]. \qquad (5)$$

Figure 1b shows that $\alpha_p(x_s)$ varies smoothly between the ice-free and ice-covered limits.

a. Budyko-style energy balance models

Budyko (1969) presented an energy balance model that is particularly tractable analytically, proposing a very simple parameterization for the divergence of the poleward heat flux:

$$\nabla\cdot\mathbf{F} = C(T - \overline{T}). \qquad (6)$$

Thus, there is a divergence of heat flux if the local temperature is higher than the global mean and convergence of heat flux if it is lower. The higher the value of C, the more efficiently heat is redistributed on the planet. Sellers (1969) also parameterized heat flux in this way but included extra model complexities that are unnecessary for present purposes.

1) TRADITIONAL ANALYSIS

An outline of the solution is briefly given here for the clarity of presentation, but it follows previous studies (e.g., Lindzen and Farrell 1977).

With this Budyko-style parameterization of the heat flux, applying Eq. (1) at the ice line ($x = x_s$) gives

$$\underbrace{\frac{Q}{4}S(x_s)(1-\alpha_s)}_{\text{absorbed shortwave}} - \underbrace{C(T_s - \overline{T})}_{\text{flux divergence}} = A + BT_s, \qquad (7)$$

where α_s is the albedo exactly at the ice line. A simple choice is to take $\alpha_s = 1/2(\alpha_1 + \alpha_2)$ (e.g., Lindzen 1990). From Eq. (7), and by construction of the model, it is seen that the OLR at the ice line is always a constant. The combination of the other two terms in the energy balance—the absorbed shortwave radiation minus the divergence of the poleward heat flux—must equal this constant.

Equation (7) can be combined with Eq. (2) to eliminate $\overline{T}$:

$$\frac{Q}{4}(1-\alpha_s)S(x_s) + \frac{Q}{4}\frac{C}{B}(1-\alpha_p) = \text{constant}. \qquad (8)$$

Substituting from (5) into (8) gives an analytical expression for $Q(x_s)$ (e.g., Lindzen 1990) that governs how the equilibrium ice line varies as a function of Q (Fig. 2a). Figures similar to Fig. 2a appear in many papers on snowball earth. Some of these studies argue on physical grounds and others provide detailed (and sometimes

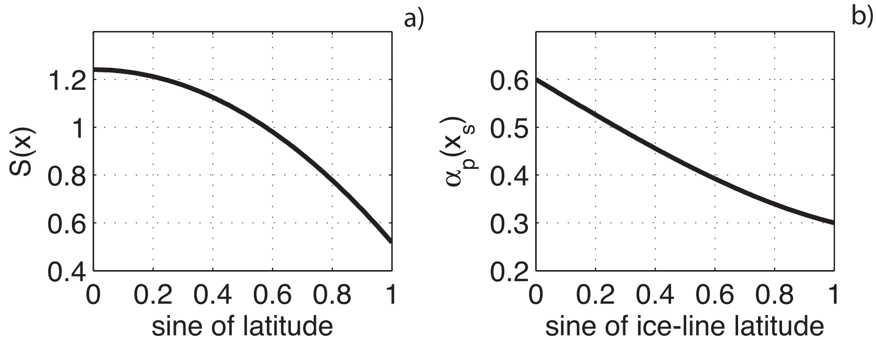

FIG. 1. (a) Normalized insolation distribution $S(x)$ as a function of latitude. The normalization is such that $\int_0^1 S(x)\,dx = 1$. (b) Planetary albedo $\alpha_p(x_s)$ as a function of the latitude of the ice line. Note that the x axes in the two panels refer to different things.

involved) mathematical proofs that no stable solution is possible when the slope of x_s versus Q is negative (e.g., Held and Suarez 1974; North 1975; Ghil 1976; Su and Hsieh 1976; Drazin and Griffel 1977; Lindzen and Farrell 1977; Cahalan and North 1979; North 1990; Shen and North 1999). The term "slope-stability theorem" has been coined to describe the proposition.

We show in the next section that a formal analysis of the ice albedo feedback provides a simple poof of the slope-stability theorem and that it gives physical insight into the cause of the instability.

2) Feedback analysis from the ice-line perspective

The instability results from the variation of albedo with changing climate state (as represented by x_s, $\overline{T}$). One way to evaluate the effect of this is to ask, what is the difference between the sensitivity of the ice line latitude to variations in the solar constant with and without albedo variations? Framing the issue in this way is at the heart of a feedback analysis (e.g., Roe 2009).

A first-order Taylor series expansion of Eq. (8) gives

$$\Delta Q\left[\frac{(1-\alpha_s)S(x_s)}{4}+\frac{1}{4}\frac{C}{B}(1-\alpha_p)\right] + \Delta x_s\left[\frac{Q(1-\alpha_s)}{4}S'(x_s)\right] - \Delta x_s\frac{QC}{4B}\alpha_p' = 0, \tag{9}$$

where the primes denote derivatives with respect to x_s. First, consider the case in which no albedo variations are permitted. In this instance, $\alpha_p' = 0$ and the sensitivity of the ice line to insolation can be written as

$$\Delta x_s = \lambda_x \Delta Q, \tag{10}$$

where

$$\lambda_x = -\frac{(1-\alpha_s)S(x_s)+\dfrac{C}{B}(1-\alpha_p)}{Q(1-\alpha_s)S'(x_s)}. \tag{11}$$

Here S' is negative and so λ_x is positive; λ_x can be straightforwardly calculated from previous expressions.

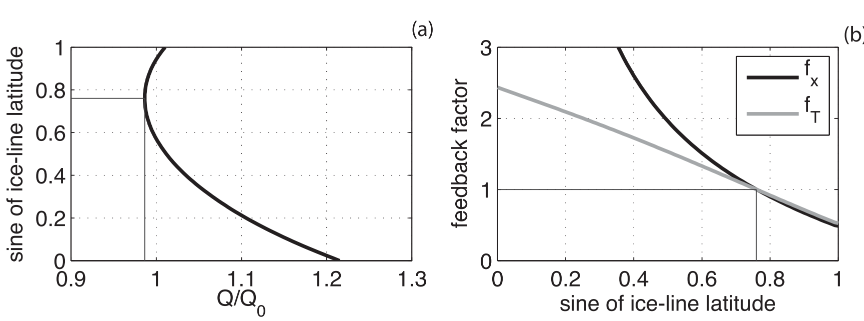

FIG. 2. Properties relating to the ice line instability in the Budyko model. (a) Equilibrium ice line as a function of insolation relative to modern. Following Lindzen and Farrell (1977), $Q_0 = 1336$ W m^{-2}: regions with positive slope are stable equilibria, negative slopes are unstable equilibria; (b) albedo feedback factors f_x and f_T. Only regions with $f < 1$ are stable equilibria.

Second, consider the case in which albedo variations are permitted. Now $\alpha_p' \neq 0$ in Eq. (9), and variations in x_s can be written as

$$\Delta x_s = \frac{\lambda_x}{1 - f_x} \Delta Q, \tag{12}$$

where

$$f_x = \frac{C\alpha_p'}{BS'(1 - \alpha_s)}. \tag{13}$$

Here f_x is the feedback factor in this problem (e.g., Roe 2009). Both α_p' and S' are negative; therefore, as expected, f_x is a positive feedback.

Catastrophe occurs in the limit $f \rightarrow 1$. Equation (13) demonstrates that, provided there is some poleward heat transport (i.e., $C \neq 0$), this instability must be present for all parameter values: since S' tends to 0 as x_s nears the equator (Fig. 1a), at some latitude f must exceed 1. The slope stability theorem also follows directly from Eq. (12): for $f_x < 1$ (i.e., stable equilibria), $\Delta x_s/\Delta Q > 0$; for $f_x > 1$ (i.e., unstable equilibria), $\Delta x_s/\Delta Q < 0$. This behavior is shown in Fig. 2b.

3) What is the physical explanation of the instability?

The mechanism of the instability can be understood physically as follows. Suppose, beginning from some equilibrium climate state, the ice line advances while Q is held constant. The higher local insolation at lower latitudes produces warming at the perturbed ice line position. Acting alone, this warming would tend to restore the ice line to its previous equilibrium position. However, the local divergence of heat flux increases at lower latitudes, and this produces cooling at the new ice line position. If the cooling is larger than the warming, then the ice line will continue to advance; therefore, the situation is unstable. We can see this from the following: the relative magnitude of these two tendencies can be found by differentiating the terms in Eq. (7) with respect to x_s and holding Q constant. Let R be the magnitude of the ratio of the cooling tendency (i.e., the increase of local heat flux divergence) to the warming tendency (i.e., the increase in local insolation); R can then be written as

$$R = \left| \frac{C \dfrac{d\overline{T}}{dx_s}\Big|_Q}{\dfrac{Q(1 - \alpha_s)}{4} \dfrac{dS}{dx_s}} \right|. \tag{14}$$

From Eq. (2), $d\overline{T}/dx_s|_Q = -Q/4B(d\alpha/dx_s)$, and so R becomes

$$R = \frac{C\alpha_p'}{B(1 - \alpha_s)S'} \equiv f_x. \tag{15}$$

Therefore, for an incremental advance of the ice line, the cooling term exceeds the warming term at the same latitude that f_x exceeds 1. Thus, we also see that the local and global perspectives on the feedback are equivalent.

The snowball instability is inevitable in this climate model simply because of the geometry of a sphere. The rate at which the local insolation increases (or in other words, the restoring warming tendency described earlier) diminishes as the ice line latitude moves equatorward (i.e., Fig. 1a) while the destabilizing effect of the local divergence of heat flux increases. As the equilibrium ice line descends to lower and lower latitudes, it becomes easier and easier for a perturbed ice line to advance. Thus, the strengthening of this positive albedo feedback as the ice line advances reflects a robust property of the climate system; therefore, it is likely to hold in more sophisticated models. We note that Lindzen and Farrell (1977, 1980), Poulsen et al. (2001), and others have explored how including dynamical circulation regimes such as the Hadley cell or additional heat transport processes, such as ocean circulation, can modify this picture and we broach this further in the discussion.

4) What is the dependency of the instability on physical parameters?

Differentiating Eq. (5) with respect to x_s and substituting it into Eq. (13) gives

$$f_x = \frac{C\alpha_p'}{B(1 - \alpha_s)S'} = \frac{C(\alpha_1 - \alpha_2)S(x_s)}{B(1 - \alpha_s)S'}. \tag{16}$$

The strength of the feedback, therefore, depends linearly on the albedo contrast between ice-covered and ice-free areas, as is perhaps intuitive. Here f_x is also proportional to C—that is, the more efficiently heat is redistributed, the stronger the feedback. In effect, this reflects that heat can be ''pulled out'' of the tropics more effectively; thereby creating a greater cooling tendency and permitting the ice line to advance more easily (see also Held and Suarez 1974). This has a strong physical basis; therefore, it is likely to also be true of models that have a more sophisticated representation of heat transport. Last, f_x is inversely proportional to B, since as noted earlier, a higher value of B means a lower sensitivity of climate to perturbations. We note that all of the model parameters enter into f at the same order, implying they have equal importance.

Setting $f_x = 1$ in Eq. (16) produces a quadratic equation for the sine of the latitude, x^*, at which the instability occurs:

$$x^{*2} - \frac{2B(1-\alpha_s)}{C(\alpha_1-\alpha_2)}x^* + \frac{2-s_2}{3s_2} = 0. \qquad (17)$$

The quadratic nature of the equation and the presence of s_2 (the coefficient in the series expansion of the insolation distribution) reflect the spherical geometry. Note the model parameters appear as the coefficient for the linear term in Eq. (17) and in the same nondimensional combination as in Eq. (16), though inverted. A decrease in this linear coefficient causes an increase in x^* (i.e., the instability occurs at a higher latitude), reflecting a less stable system. Following the arguments of the previous section, the latitude of the instability is also the latitude of the ice line at which the net incoming energy fluxes are independent of x_s: equatorward of this latitude, an advance of the ice line leads to a net cooling at the ice line; poleward of this latitude, an advance of the ice line leads to a net warming at the ice line.

5) Feedback analysis from the global temperature perspective

The magnitude of a feedback within a system can depend on the variable or field of interest (e.g., Roe 2009). This can be illustrated by recasting the EBM system to solve for global mean temperature instead of ice line latitude. This makes the problem closer to the normal definition of the climate sensitivity to a radiative perturbation (e.g., Charney et al. 1979; Knutti and Hegerl 2008; Roe 2009).

Now we solve for changes in $\overline{T}$ due to changes in Q. First, suppose again that there is no albedo feedback (i.e., $\alpha_p' = 0$). In this case, from Eq. (2), first-order perturbations in temperature and solar constant are related by

$$\Delta\overline{T} = \lambda_T \Delta Q, \qquad (18)$$

where

$$\lambda_T = \frac{(1-\alpha_p)}{4B}. \qquad (19)$$

This is the equivalent of the standard climate sensitivity parameter for this problem (e.g., Roe 2009); however, in this case it is the sensitivity to changes in solar constant, not to imposed independent forcing due to CO_2. Here λ_T^{-1} measures the basic stabilizing tendency in the energy balance of the planet, whereby the outgoing longwave radiation acts to restore temperatures to equilibrium after a perturbation. For a given change in insolation, a higher value of λ_T^{-1} means a smaller temperature change and therefore reflects a stronger damping tendency. Defined in this way, climate sensitivity decreases in a colder climate because as the planetary albedo increases, a given increment in insolation produces less radiative forcing in terms of what is actually absorbed.

Now if instead the albedo is allowed to vary with temperature, the right-hand side of the equation must include the additional radiative perturbation that occurs in response to the change in albedo:

$$\Delta\overline{T} = \lambda_T \Delta Q - \frac{Q}{4B}\frac{d\alpha_p}{d\overline{T}}\Delta\overline{T} \quad \text{and} \qquad (20)$$

$$= \lambda_T \Delta Q - \frac{Q}{4B}\alpha_p' \frac{\Delta x_s}{\Delta\overline{T}}\Delta\overline{T}. \qquad (21)$$

This last term on the right-hand side is the albedo feedback. Solving for $\Delta\overline{T}$ explicitly gives

$$\Delta\overline{T} = \frac{\lambda_T}{1-f_T}\Delta Q, \qquad (22)$$

where f_T is the albedo feedback factor (e.g., Roe 2009) and is given by

$$f_T = -\frac{Q}{4B}\alpha_p' \frac{\Delta x_s}{\Delta\overline{T}} = -\frac{\frac{Q}{4}\alpha_p'}{B\frac{\Delta\overline{T}}{\Delta x_s}}, \qquad (23)$$

where the Δ notation means that the derivative is calculated along the curve $x_s = x_s(Q, \alpha_p')$ calculated from Eq. (12).

As with any positive feedback, Eq. (23) reflects competing tendencies on a conservation equation (e.g., Roe 2009). In this case, the numerator on the right-hand side reflects the destabilizing process of the albedo increasing as the ice line advances equatorward, and the denominator reflects the stabilizing process of changes in the longwave radiation to space. Equation (23) is quite general and could readily be diagnosed from perturbation experiments using global climate models, for example. The relationship between the ice line feedback and the global temperature feedback comes from the following equation:

$$\frac{\Delta\overline{T}}{\Delta Q} = \frac{\partial\overline{T}}{\partial\alpha_p}\alpha_p' \frac{\Delta x_s}{\Delta Q} + \left.\frac{\partial\overline{T}}{\partial Q}\right|_{\alpha_p=\mathrm{const}}, \qquad (24)$$

which can be rewritten as

$$\frac{\lambda_T}{1-f_T} = -\frac{Q\alpha_p'}{4B}\left(\frac{\lambda_x}{1-f_x}\right) + \lambda_T. \qquad (25)$$

From this equation, it is straightforward to demonstrate that f_x and f_T both cross 1 at the same ice line latitude, shown in Fig. 2b.

b. Diffusive energy balance models

North (1975) suggested an alternative, and arguably somewhat more physical, parameterization for the poleward heat flux, proposing that it be parameterized as proportional to the local meridional temperature gradient. In this case $\nabla \cdot \mathbf{F}$ in Eq. (1) is given by

$$\nabla \cdot \mathbf{F} = -D\frac{d}{dx}(1-x^2)\frac{dT}{dx}. \tag{26}$$

1) Traditional analysis

North (1975) demonstrated that an accurate analytical approximation to Eqs. (1) and (26) could be obtained using hypergeometrical functions and matching boundary conditions at the ice line. North (1975), Cahalan and North (1979), Shen and North (1999), and others have studied the stability properties of these solutions, analyzing the time-dependent behavior of perturbations away from the derived equilibrium solutions.

Figure 3a reproduces the original analytical solutions derived by North (1975), using his chosen parameter set (which are slightly different from those used up to this point in this paper). From the slope of x_s versus Q, it is clear that stable climates do not exist equatorward of $x_s \approx 0.6$. In addition, there is also a striking phenomenon poleward of $x_s \approx 0.95$, the so-called small ice cap instability (e.g., North 1984): beyond some latitude, the slope of x_s versus Q turns negative, implying that the polar ice cap can only be stable if it extends past some finite latitude. The reasons for this behavior has been analyzed in detail in simple systems (e.g., Lindzen and Farrell 1977; North 1984), though its presence in more complete climate models is still discussed (e.g., Crowley et al. 1994; Lee and North 1995; Langen and Alexeev 2004; Rose and Marshall 2009; Enderton and Marshall 2009).

2) Feedback analysis

A simple alternative to the time-dependent analyses cited earlier is to calculate the feedback strengths by direct substitution of the analytical solutions provided in North (1975) into Eqs. (18), (23), and (25). Figure 3d shows both f_x and f_T; f_x behaves as expected—it lies between 0 and 1 in the stable ice line regime and exceeds 1 for unstable ice line regimes. The behavior of f_T is more interesting. It goes through two singularities and actually becomes negative near the equator and near the pole.

The cause of this peculiar behavior is related to the small ice cap instability and, as it turns out, there is a directly analogous counterpart near the equator. The explanation closely follows arguments in Lindzen and Farrell (1977) for the small ice cap instability and is illustrated schematically in Fig. 3. Three curves are shown for equilibrium climate states using the Budyko-style approximation for $\nabla \cdot \mathbf{F}$, but using different values for the ice line albedo [(i) $\alpha_s = \alpha_1$; (ii) $\alpha_s = 0.5 \times (\alpha_1 + \alpha_2)$ as has been used up to now; and (iii) $\alpha_s = \alpha_2$]. Recall that these curves give pairs of (x_s, Q) that are equilibrium solutions of the model equations, and that the stability of these equilibrium states can be judged from whether $dx_s/dQ > 0$ (stable) or $dx_s/dQ < 0$ (unstable).

The small ice cap instability can be understood by considering the intersection of these curves with $x_s = 1$. Imagine starting with an ice-free earth and high Q (point A_1 in Fig. 4). If Q is now gradually lowered, then the system moves toward point A_2. As soon as any ice forms on the planet though, the solution trajectory must jump from A_2 to A_3 because of the discontinuity in albedo (i.e., $\alpha = \alpha_s$ at this point). In other words, the introduction of any ice at all means somewhat counterintuitively, that the solar constant is no longer large enough to maintain the ice at that latitude in equilibrium.

As pointed out by Lindzen and Farrell (1977), in the Budyko-style EBM the nonlocal nature of the heat transport means the discontinuity is confined to $x_s = 1$. For North-style diffusion however, the influence of the albedo discontinuity leads to a boundary layer that extends into the domain with a characteristic length scale equal to $\sqrt{D/B}$ (see also North 1984). The thick curve shows the trajectory of equilibrium (though unstable) states from A2 to A3. These are achieved by increasing Q and T (Fig. 2b), even though the ice line is descending equatorward. Thus, the gradient $\Delta\overline{T}/\Delta x_s$ is negative (Fig. 3c) and so from Eq. (23), f_T is also negative.

There is a directly analogous discontinuity at the equator. Start with an ice-covered earth and low Q (point B_1). If Q is now gradually increased, then the system moves along the path from B_1 to B_2. But again, as soon as any ice-fee areas emerge, the solution trajectory must jump to B_3. Following the same reasoning as before, $\Delta\overline{T}/\Delta x_s$ reverses (Fig. 3c) and so f_T is negative. The thick green line in Fig. 3c also indicates schematically the penetration of the effect of this discontinuity into the domain for North-style diffusive transport. The equatorial discontinuity is not readily apparent in the x_s versus Q curves because the slope of the curve from B_2 to B_3 has the same sense as the slope of dx_s/dQ at slightly higher latitudes. Taken together, the polar and the near-equator instabilities produce the thick green curve in Fig. 4, which is similar to the curve of x_s versus Q curve in Fig. 3a.

In summary, imagine a global temperature increase from an unspecified cause. For most values of x_s, this causes a retreat of the ice line, amplifying the original

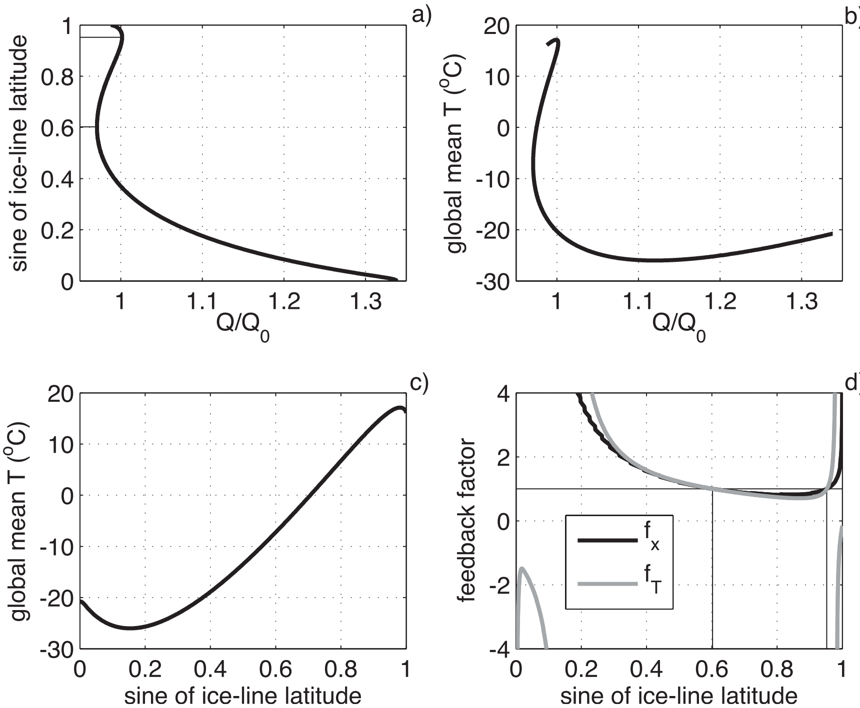

FIG. 3. Properties of solutions to the North diffusive EBM. (a) The ice line as a function of Q/Q_0, using North (1975) analytical solutions and parameters. The thin lines show turning points. (b) Global mean temperature vs solar constant for the same solution. (c) Global mean temperature vs ice line for the same solution. (d) Albedo feedback factors f_x and f_T. The thin lines confirm that the feedbacks exceed 1 at the latitude of the turning points in (a). Note that f_T becomes negative near the equator and the pole. Refer to text for an explanation.

warming [Fig. 3c and Eq. (20)]. However, in the vicinity of the equator and pole, the discontinuity in albedo exerts a stronger control on the system dynamics, and the warming is in fact associated with an advance of the ice line. This damps the original warming and so the feedback is negative. Although this only occurs here in equilibrium climate states that are unstable, it is an exotic illustration of the point that if the dominant physical processes change as a function of mean climate state, then the magnitude and even the sign of the feedback can vary (e.g., Roe 2009).

3. Discussion

In essence, the analysis presented here recasts existing solutions for simple energy balance models into the language of feedback analysis. In doing so, the physical cause of the snowball earth instability can be clearly and simply laid out. From the perspective of the global energy balance, the strength of the feedback is determined by the competition between the stabilizing tendency of the outgoing longwave radiation and the destabilizing tendency of less radiation being absorbed as the planet brightens. From the perspective of the ice line, the feedback is the ratio of changes in local insolation and in the divergence of the poleward heat flux as x_s changes.

Our analysis enables derivation of simple expressions for the strength of the albedo feedback as a function of mean climate state and choice of climate parameters. One principal control is, of course, the spherical geometry of the earth, which, at least within the strictures of these simple models, makes the instability inevitable at some latitude. In the case of the Budyko-style model, the latitude of the ice line instability also depends on a simple nondimensional combination of model parameters.

We have investigated the apparently strange result that, for diffusive parameterizations of heat flux, the ice albedo can even act as a negative feedback (i.e., have a stabilizing effect) on global temperature variations. It happens here only for climate states that are unstable because of the very tight coupling assumed between the ice line and temperature in the energy balance model. However, the result that global mean temperature might have a minimum at a nonzero ice-line latitudes because of the albedo discontinuity is quite physical. It remains to be explored whether this negative ice albedo

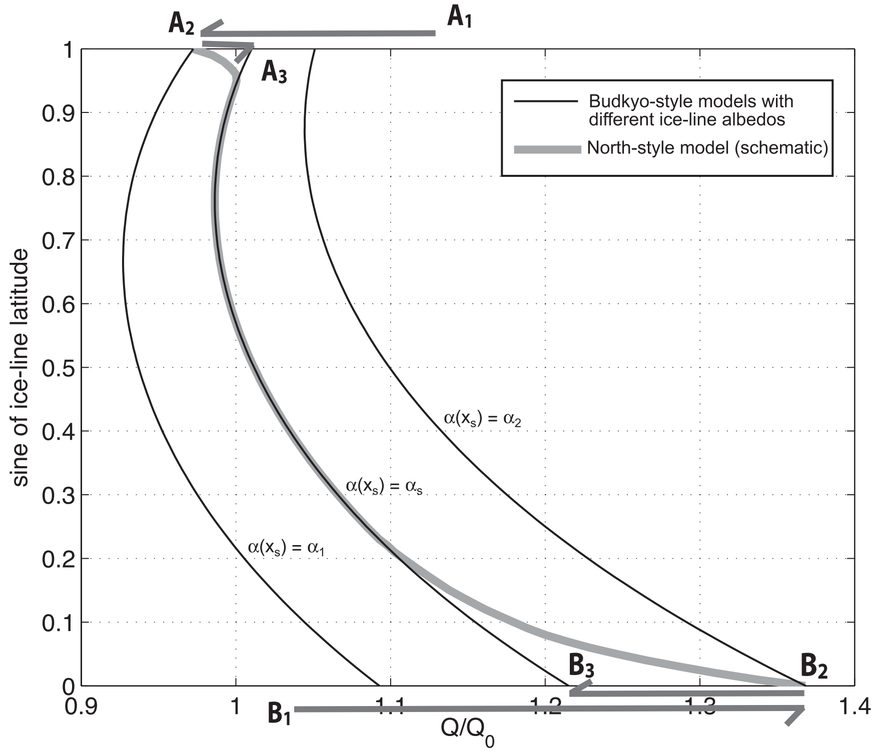

FIG. 4. Schematic explanation of small ice cap instability, and the regions of negative f_T feedback, extending the arguments of Lindzen and Farrell (1977). Refer to text for details.

feedback is just a curiosity of these particular models, or if it can help explain the occurrence of equilibrium ''slush ball'' states (i.e., an ice-free equatorial band) found in some climate models (e.g., Hyde et al. 2000; Crowley et al. 2001) and that have been argued to be more consistent with geological evidence (Allen and Etienne 2008). Another useful diagnostic is suggested by the results in sections 2a(4) and 2a(3). When the overall climate is stable, it is because an equatorward advance of the ice line causes a net warming at the ice line. This is likely a very general result. Studying the energy budget response to an ice line perturbation in models that exhibit slush-ball states would elucidate which terms are responsible for that warming, and perhaps therefore explain the differences from models that do not exhibit slush-ball states.

The very concept of a feedback implicitly partitions the system into a reference state and a set of physical ''feedback'' processes (e.g., Roe 2009). In this context, having an ice albedo feedback means introducing a process that allows the albedo to vary with a climate state. A straightforward lesson that also applies to more complex systems is that the effect of adding this process depends on which part of the system is of interest. In this simple case studied here, the feedback strength is different for the global mean temperature and for the ice line.

Our representations of the feedbacks by ratios of derivatives illustrate the general feature of feedbacks; whereas in the simplified physical system considered in this paper, the derivatives were taken with respect to the spatial variable x_s, the primes could more generally indicate derivatives taken with respect to other climate variables, such as circulation pattern, atmospheric composition, among others.

The zonal mean, annual mean EBMs presented here are obviously highly idealized representations of the real world. Severe approximations have been made in their derivation, not the least of which are the absence of clouds and a seasonal cycle, and these approximations render the albedo feedback as being substantially larger than is inferred from GCMs for the modern climate (e.g., Soden and Held 2006). It would be of interest to diagnose ice albedo feedbacks within GCMs as the solar constant is reduced (following, e.g., the methods of Soden and Held 2006), and to evaluate if the feedback strength varies in ways that are consistent with the predictions from (16).

Some studies have suggested that there might be a ''stability ledge'' because of the effects of the Hadley cell (Lindzen and Farrell 1977); some climate model results suggests that ocean transports (Poulsen et al. 2001) or latent heat fluxes (Poulsen 2003) can act to inhibit a complete glaciation. These processes could, in principal, be cast as additional feedbacks in the energy budget. To first order, the net effect on the climate is given by the sum of the individual feedback factors and so isolating just the ice albedo feedback provides a guide for how strong those negative feedbacks have to be to create a stable equilibrium (i.e., the sum must be less than one).

Recent advances in feedback analysis permit the full spatial structure of climate feedbacks to be calculated (e.g., Soden et al. 2008), and it can even include ocean heat uptake (Gregory and Forster 2008). A full feedback diagnosis of the simulations from more complicated models such as Voigt and Marotzke (2009) would permit the relative importance of individual processes in these models (and the uncertainties in them) to be propagated through the system dynamics. One important and robust expectation is that uncertainties in physical process (and in model parameterizations of them) lead to large uncertainties in the system response in the vicinity of $f = 1$ because of the strong amplification that is occurring (e.g., Roe 2009).

The snowball earth phenomenon illustrates how localized physical processes can have a global effect. Here, strong model assumptions control how something happening at one particular latitude—the albedo changing because of an ice line advance—acts to affect the global mean climate. In nature other important feedbacks are also localized, such as the strong negative feedback of subtropical stratus decks (e.g., Sanderson et al. 2008), or the high-latitude deep-ocean heat uptake (e.g., Gregory and Forster 2008; Winton et al. 2010; Baker and Roe 2009). Perhaps one important way forward for improving both global and regional climate predictions will be to better understand how these regional processes combine to give the full, global system response.

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