NOTES AND CORRESPONDENCE

Baroclinic Adjustment in a Two-Level Model with Barotropic Shear

Gerard H. Roe and Richard S. Lindzen

Center for Meteorology and Physical Oceanography, Massachusetts Institute of Technology, Cambridge, Massachusetts

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ABSTRACT

Baroclinic instability in two-level models is characterized by a critical vertical shear, for values above which the flow is unstable. Existing studies of nonlinear baroclinic equilibration in two-level models suggest that, while equilibration does occur, it does so for values of vertical shear that are supercritical. The criterion used for the critical shear, however, ignores nonlinear changes in barotropic (meridional) shear. The present note estimates, using the two-scale formalism, the effect of both jet scale and damping on the critical value of vertical shear. The results suggest the barotropic shear in the equilibrated states may be sufficient, in the presence of damping, to render the equilibrated states neutral. More generally, it appears important to take account of the nature of the evolved flow when assessing the stability properties of the equilibrated state.

1. Introduction

The notion that hydrodynamic instabilities act to bring a flow toward a state that is neutral with respect to the instability has long been a part of fluid mechanics. It receives its most common application in the form of convective adjustment. Not surprisingly, there have also been attempts to use this notion in problems involving baroclinic instability. The problem here is the absence of a simple well-defined criterion for stability. For example, the classic Charney and Eady problems do not have a critical shear. Moreover, the only theoretical statement concerning instability is the Charney–Stern theorem, which states that a necessary but not sufficient condition for instability is that there exists some surface where the meridional gradient of a quantity related to potential vorticity changes sign. In most realistic cases this criterion is met due to a delta-function contribution to the gradient associated with meridional surface temperature gradients. Neutralization is most easily achieved by eliminating surface temperature gradients and smoothly merging to the interior gradient. Unfortunately, the resulting state looks nothing at all like the observed climatological state. Recently, Lindzen (1993) noted that it was possible to neutralize a flow that still satisfied the Charney–Stern condition. He noted that if one mixed potential vorticity (PV) within the troposphere, one could, by raising the tropopause (defined as the level where the PV gradient is concentrated) and narrowing the subtropical jet, produce baroclinic neutrality by producing a short-wave cutoff that was smaller than zonal wavenumber 1. The question of whether the nonlinear dynamics of the atmosphere act to approach this state is, however, a difficult one to resolve insofar as tropopause height, jet width, and jet position are all involved.

The situation is potentially simpler for two-level models (Phillips 1954). A two-level model does have a critical shear (which is directly derivable from the Charney–Stern condition; viz Lindzen 1990). Although the two-level model is not an appropriate approximation to the atmosphere, it has been suggested that its neutral state might provide an estimate of the observed midtropospheric meridional temperature gradient in the atmosphere (Stone 1978). Apart from such speculations, the nonlinear behavior of two-level models is easier to compute than the nonlinear behavior in realistic continuous atmospheres. Such calculations can be used to see if, at least in two-level models, an equilibrated state develops that is approximately neutral. This might lend some confidence to the notion that baroclinic adjustment is relevant to the atmosphere. Such attempts were made by Cehelsky and Tung (1991) and Stone and Branscome (1992). Both studies did find equilibrated states that parametrically behaved like baroclinic adjustment suggested, but where the shears were significantly larger than the purported neutral values (i.e., supercritical). It was suggested by both

Corresponding author address: Dr. Richard S. Lindzen, Center for Meteorology and Physical Oceanography, Massachusetts Institute of Technology, Room 54-1720, Cambridge, MA 02139.
E-mail: lindzen@wind.mit.edu

studies that wave–wave interactions led to longer dominant waves whose critical shear was larger than that which pertained to the most unstable mode. Such a result is not, on the whole, encouraging for the straightforward application of baroclinic adjustment in a more realistic situation.

However, Stone and Branscome (1992) calculated the critical shear assuming no meridional (barotropic) shear, whereas Cehelsky and Tung (1991) calculated the instability of modes with respect to the initial state rather than the equilibrated, final state. Their nonlinearly equilibrated flows did display barotropic shears, and as shown by Ioannou and Lindzen (1986) and James (1987), such shears do affect instability. Whereas James (1987) was concerned with the growth rates of the unstable modes in a set up similar to the one adopted below, the more pertinent parameter for the baroclinic adjustment hypothesis is the critical shear. The purpose of the present note is to assess the degree to which the barotropic shear of the subtropical jet affects the critical shear. This note is not meant to be an exhaustive calculation of the stability profiles for the atmosphere; given the approximations inherent in the two-level model, the point of such a calculation is questionable. Since the intention is merely to explore the possibility that the equilibrated state might be baroclinically neutral in the conventional linear sense, we choose to adopt the simple two-scaling approach of Ioannou and Lindzen (1986) and apply it to the two-level model. It is found that the critical shear is indeed dependent on the narrowness of the jet. For jets with scales about equal to those observed in the atmosphere, the critical shear is quite sensitive to small changes in the jet width.

2. The equations

The problem addresses the stability of the basic state that is a function of height z and latitude y to perturbations of the form $\exp[ik(x - ct)]$ (k is the zonal wavenumber and c is the phase speed, which is in general complex). Denoting perturbation quantities by primes, the vorticity and thermodynamic equations may be respectively written as

$$ik[\bar{u}(y, z) - c]\left\{-k^2\phi' + \frac{\partial^2\phi'}{\partial y^2}\right\} + ik\phi'\left[\frac{\partial^2}{\partial y^2}\bar{u}(y, z) + \beta\right] - f_0^2\frac{\partial w'}{\partial z} = 0 \quad (1)$$

$$ik[\bar{u}(y, z) - c]\frac{\partial\phi'}{\partial z} - ik\phi'\frac{\partial}{\partial z}\bar{u}(y, z) + w'N^2 = 0, \quad (2)$$

where $\bar{u}(y, z)$ is the basic-state zonal wind, ϕ is the geopotential height, w is the vertical velocity, f_0 is the Coriolis parameter at a given latitude, and N is the Brunt–Väisälä frequency. Subsequently, the primes will be dropped from the perturbation fields.

In the instance of a two-level model with no meridional variation, the vorticity equation is evaluated in the middle of each of the two layers and the thermodynamic equation is applied at the interface between the two layers. Centered differences are used to calculate the derivatives, and boundary conditions of zero vertical velocity are applied at the upper and lower lids. The three resulting equations can then be solved for the phase speed c. Details may be found in Lindzen (1990).

If there is meridional variation allowed in the basic state, the problem becomes inseparable in height and latitude. Ioannou and Lindzen (1986, 1990) tackled the problem by using an assumption that the meridional scale of the jet is large compared with the deformation radius. The meridional variation of the jet is given in terms of the slow variable Y, where $Y = y/L$ and $L \gg 1$; Y and y are then treated as independent variables. Thus, L is a measure of how many times broader the jet is than the radius of deformation. Typical values in the atmosphere range between about 0.8 and 1.7 (Ioannou and Lindzen 1986). Although the scaling is strictly valid in the limit only of large L, the precise determination of the proper value at which L is large is ambiguous. Lin and Pierehumbert (1988), employing a full numerical calculation, find good agreement with Ioannou and Lindzen's (1986) results even at jet widths appropriate to the atmosphere, suggesting that our results are in the domain of asymptotic validity. Physically, this is a result of the fact that the "wave geometry,"—the configuration of the turning points and critical levels within the fluid domain—does not change as the jet tightens. The wave geometry essentially confines the modes to a waveguide in the core of the jet, and the first-order approximation of the two-scaling approach reflects this behavior, provided that the mode remains wavelike in the meridional direction within the jet.

So, a separable solution is sought of the form

$$\phi = W(y)\left[\tilde{\phi}_0(Y, z) + \frac{1}{L}\tilde{\phi}_1(Y, z) + \frac{1}{L^2}\tilde{\phi}_2(Y, z) + \cdots\right]$$

and

$$c = c_0 + \frac{1}{L}c_1 + \frac{1}{L^2}c_2 + \cdots.$$

Upon substitution, Eq. (1) and Eq. (2) become to first order

$$-k^2 + \frac{\beta}{\bar{u}(Y, z)} - \frac{f_0^2 \frac{\partial w}{\partial z}}{ik[\bar{u}(Y, z) - c_0]\tilde{\phi}_0} = \frac{\frac{d^2}{dy^2} W(y)}{W(y)} \tag{3}$$

$$ik[\bar{u}(Y, z) - c] \frac{\partial \tilde{\phi}_0}{\partial z} - ik\tilde{\phi}_0 \frac{\partial}{\partial z} \bar{u}(Y, z) + w'N^2 = 0. \tag{4}$$

Because the lhs of Eq. (3) is independent of y and the rhs is independent of z, the equation may be separated into two equations linked by a variable dependent only on Y:

$$-k^2 + \frac{\beta}{\bar{u}(Y, z)} - \frac{f_0^2 \frac{\partial w}{\partial z}}{ik(\bar{u}(Y, z) - c_0)\tilde{\phi}_0} = l^2(Y) \tag{5}$$

and

$$\frac{d^2}{dy^2} W(y) + l^2(Y)W(y) = 0$$

or

$$\frac{d^2}{dY^2} W(Y) + L^2 l^2(Y)W(Y) = 0. \tag{6}$$

The problem is now only loosely coupled in y and z through the variable $l(Y)$. Equations (5) and (6) may be applied to the two-level fluid in the same way as for the homogenous (no y-variation) problem, yielding the same dispersion relation except that k^2 is replaced by $d^2(Y) = k^2 + l^2(Y)$. To make things clearer, the equations may be nondimensionalized:

$$z = H\tilde{z}$$

$$\left(z \frac{\partial \bar{u}}{\partial z}, c\right) = mH\left(\frac{\partial \tilde{\bar{u}}}{\partial \tilde{z}}, \tilde{c}\right)$$

$$(x, y) = a(\tilde{x}, \tilde{y}) = \frac{NH}{f_0}(\tilde{x}, \tilde{y}),$$

where a is the radius of deformation and m is the magnitude of the shear at the center of the jet (assumed constant with height). The dispersion relation is then given by (dropping the tildes)

$$c = u_M(Y) + \left[-\frac{r}{d(Y)^2}\left(\frac{1 + d(Y)^2}{2 + d(Y)^2}\right)\right] \pm \left\{\frac{r^2}{d(Y)^4(2 + d(Y)^2)^2} - u_T^2(Y)\left[\frac{2 - d(Y)^2}{2 + d(Y)^2}\right]\right\}^{1/2}, \tag{7}$$

where u_M is the mean, and u_T is half of the difference, of the winds in the two levels. $r = (\beta H)/(\epsilon m)$ with $\epsilon = f_0^2/N^2$. Note that r is thus the nondimensional measure of the inverse of the shear.

The critical shear for uniform flow (no jet) in a channel of width ΔY may be deduced from Eq. (7). In this case u_M, u_T, and d are no longer functions of Y and, because of the nondimensionalization of the equations, are fixed; $u_M = 0.5$, $u_T = 0.25$, and, for the lowest meridional mode, $d^2 = k^2 + (\pi/\Delta Y)^2$. A mode will be unstable (i.e., imaginary part of c greater than zero) if the discriminant in Eq. (7) is less than zero. The strength of the shear of the basic-state varies inversely with the parameter r. It is readily shown (e.g., Lindzen 1990) that for r less than one, there are unstable modes, whereas for r greater than one, there are none. If there is a jet, then d is, in general, complex and the value of the critical shear needs to be calculated. In the limiting case of a very large-scale jet, the answer must, of course, tend to the critical shear given by $r = 1$.

The problem has been reduced to inverting the traditional two-level model dispersion relationship at each latitude subject to Eq. (6) and the appropriate boundary conditions. It can be solved by applying a "shooting" method across the domain and iterating to a converged solution as outlined in Ioannou and Lindzen (1986).

3. Results

For the results presented below a jet structure of the following form was used:

$$u = \frac{mz}{\sqrt{1 + Y^2}}.$$

The boundary conditions require that the solution must be evanescent at large values of Y. Since a growing mode is largely confined to the jet region, the results for jet scales appropriate to the atmosphere will not depend on whether an open domain or a channel of around 5000 km is used as in, for example, Stone and Branscome (1992).

To facilitate comparison with Stone and Branscome (1992), we adopt their values of $N^2 = 1.35 \times 10^{-4}$ s^{-2}, $f_0 = 9.35 \times 10^{-5}$ s^{-1}, and $H = 7.5$ km. The critical shear for a two-level model of this construction and with no y-variation is 2.02 m s^{-1} km^{-1}. Stone and Branscome chose the shear of their basic state to be about 4.7 m s^{-1} km^{-1}, equivalent to a maximum jet at the tropopause of 70 m s^{-1}. In their standard run, they also include Ekman friction acting on the surface winds with a timescale of 2.5 days and a radiative damping time of 25 days.

The most straightforward manner to introduce damping in the present formulation is to apply the same linear damping to the momentum and thermodynamic equations. Consistent with the large L assumption, this can be implemented just making the transformation

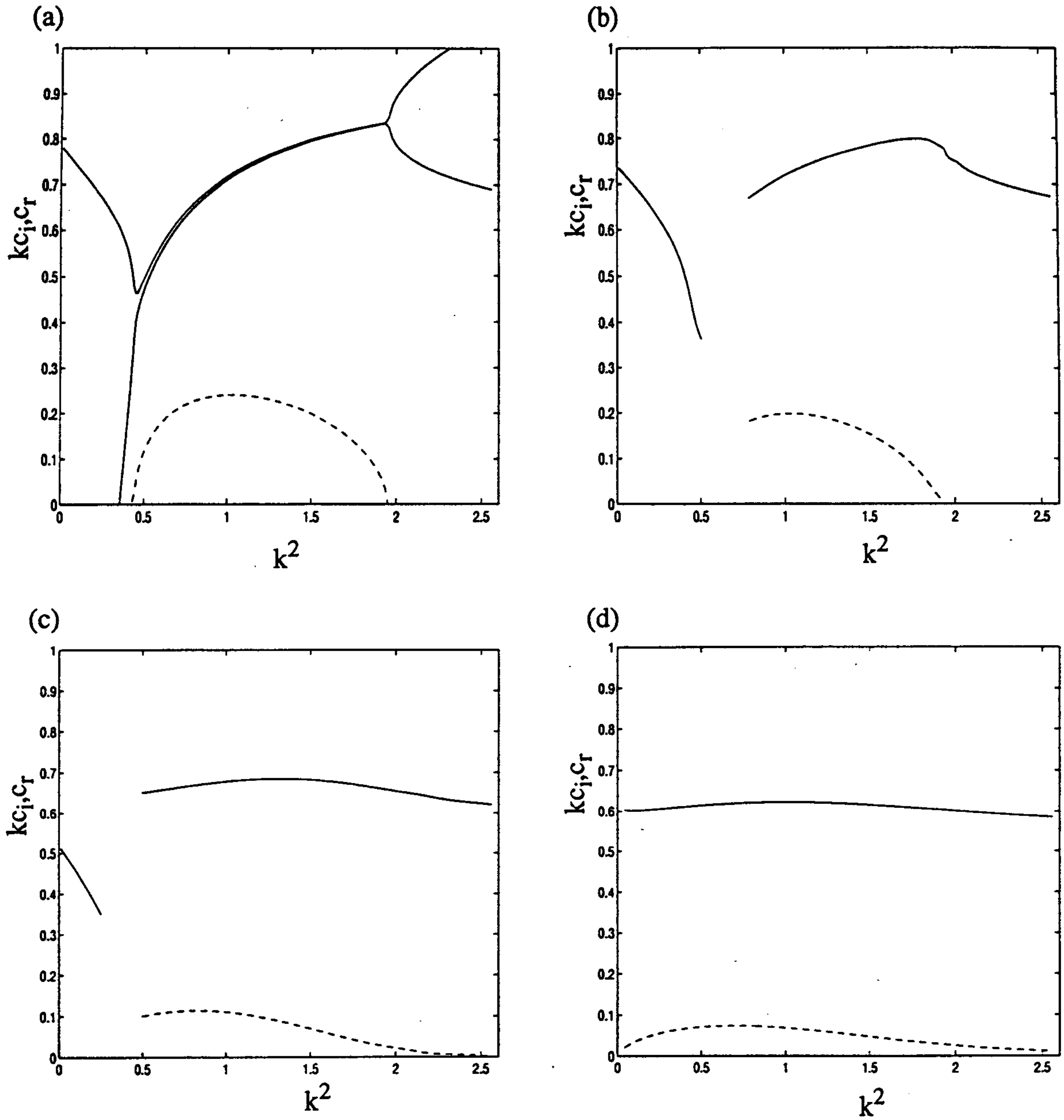

FIG. 1. Dispersion relations for different jet structures. Here, k is the zonal wavenumber, c_r (plotted as a solid line) is the real part of the phase speed, c_i is the imaginary part of the phase speed, and hence kc_i (plotted as a dashed line) is the growth rate; (a) has no jet, (b) has $L = 10$, (c) has $L = 1.5$, and (d) has $L = 0.75$. All plots are for $r = 0.43$ and damping time $\alpha = 0.02$. Axis scales are nondimensional.

$U \rightarrow U - i(\alpha/k)$, where α is the inverse of the damping timescale. It is interesting that the results presented below are somewhat sensitive to the damping timescale adopted, whereas Stone and Branscome's results are rather insensitive. This may perhaps be due to the different formulations of the dissipation. It is not obvious what linear damping timescale corresponds most closely to an Ekman damping timescale, so we present results for a variety of damping times. The appropriateness of either formulation applied to the two-level model is questionable.

The dispersion relations for different jet widths are shown in Fig. 1. In the region of the long-wave cutoff, the algorithm described did not converge well, so we show only wavenumbers for which the solution was strongly convergent. The convergence was poor because of the presence of a singularity in the inversion of the dispersion relation. Specifically, the shooting method used started to jump between the two branches of the inverted dispersion relation. This is characteristic of such methods when applied to the most weakly growing modes. However, it is the critical shear that is of interest, and the solution at the maximally growing wavenumber was well converged.

The solution for a broad jet agrees well with the no-jet solution, as would be expected (see Fig. 1). As the jet narrows, a marked reduction in growth rates occurs. Such a reduction implies that the critical shear for instability may be increasing. Figure 1 also shows that the narrower the jet, the longer the wavelength of the most unstable mode. The variation of growth rate with shear may be calculated for jets of different widths, and the results for a strong damping timescale of 2.5 days are shown in Fig. 2. For jets with meridional scales comparable to those observed in the atmosphere, the critical shear may be a factor of 2 larger than the critical shear in the absence of both a jet and damping.

The sensitivity of the critical shear to the jet scale increases as the jet tightens, so that even relatively small changes in structure may affect the critical shear. The variation of the critical shear with jet width is shown in Fig. 3 for several different frictional timescales. Quantitative comparisons of these results with previous studies may be a little deceptive; the exact value of the critical shear will depend on the setup of the model and in particular on the magnitude and form of dissipation adopted.

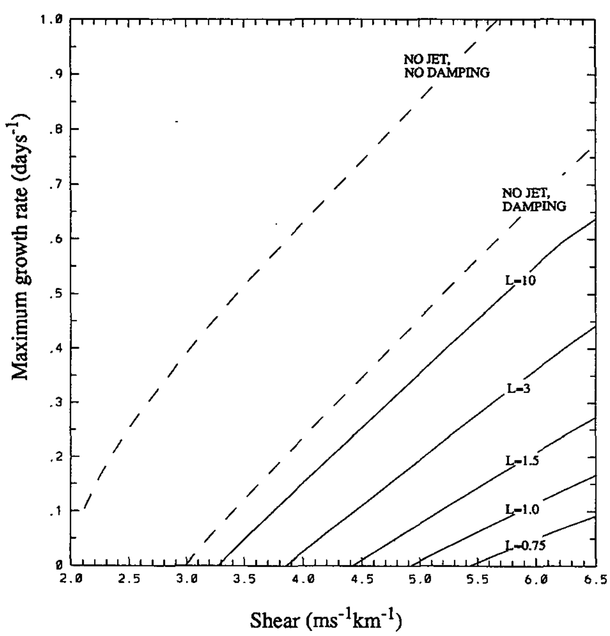

FIG. 2. Maximum growth rate vs basic-state shear for various jet widths. Damping, where applied, is 2.5 days. The critical shear for a given jet scale is given by the intersection of the appropriate curve and the shear axis. In calculations without a jet, the shear is independent of y and equal in magnitude to the maximum shear of the jet (i.e., at the center).

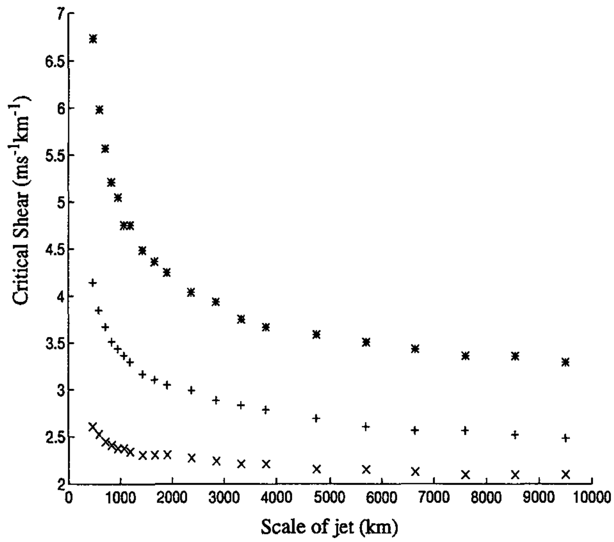

FIG. 3. Critical shear vs jet scale for different damping timescales. The cross symbols are results for damping of 10 days, the plus symbols for 5 days, and the asterisk symbols for 2.5 days. The scale of the jet is equal to the Rossby radius multiplied by L.

4. Summary and conclusions

The presence of a barotropic jet stabilizes a flow by confining the region of strongest shear. We have shown that, for jets of atmospheric scale, the effect of this stabilization on the critical shear in the two-level model is striking. Dependent on the dissipation used, we find that the shear necessary for instability may be a factor of 2 or so larger than for the case of a two-level model with no jet.

We also find that the sensitivity of the critical shear to small variations in the jet structure is acute, which suggests that it is important in nonlinear studies to assess the stability of the flow, specifically the degree to which the flow exceeds the critical shear, against the equilibrated final state rather than the initial state. In particular, it would appear that the equilibrated states in Cehelsky and Tung (1991) and Stone and Branscome (1992) may well be near neutral, but that the critical shear depends on the nature of the evolved flow and cannot be determined a priori. This is true of any search for equilibration and is not restricted to two-level models.

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