

What do glaciers tell us about climate change?

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Abstract

Glaciers are pretty and move around a lot on the landscape due to the random stochastic climate forcing inherent to a constant climate. They obviously also respond to changes in climate. The tricky part is whether, in absence of direct instrumental measurements, climate changes can be inferred from variations in glacier length.

1 Introduction

Some good questions:

- How is the distribution of maximum excursions in glacier length related to the precipitation and temperature fluctuations?

- under what conditions can we conclude that an observed glacier retreat is an indicator of an actual climate change, rather than just what might be expected in the ordinary course of events as a natural fluctuation in response to interannual climate variability?

- are some glaciers better than others as indicators of climatic warming?

- how does observing the retreat of glaciers in more than one location change the interpretation of the cause?

2 A simple glacier model

Roe and O'Neal (2008) derive an expression for the time evolution of a linear glacier model:

$$\frac{dL'(t)}{dt} + \frac{\mu A_{abl} \Gamma \tan \phi}{wH} L'(t) = \frac{A_{tot}}{wH} P'(t) - \frac{\mu A_{T>0}}{wH} T'(t), \quad (1)$$

where L' is the perturbation from the equilibrium length, L_{eq} ; ϕ is the basal slope of the glacier; w is the width of the glacier tongue; H is the characteristic thickness. The A s denotes different equilibrium areas on the glacier: A_{tot} is the total area; A_{abl} is the ablation area; $A_{T>0}$ is the area over which the melt season temperature is above 0 °C. μ is the melt rate in m yr^{-1} , per degree of melt-season temperature. P' is the perturbation in annual accumulation rate, and T' is the annual

perturbation in melt-season temperature.

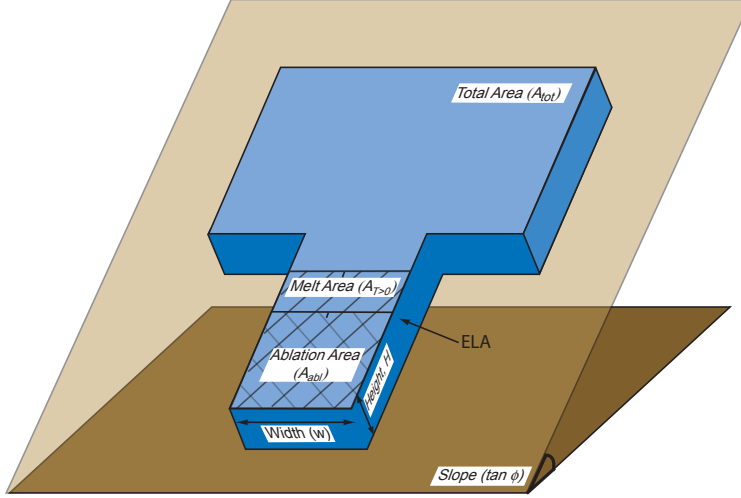


Figure 1: Idealized geometry of the linear glacier model, based on Johanneson et. al. (1989). Precipitation falls over the entire surface of the glacier, A_{tot} , while melt occurs only on the melt-zone area, $A_{T>0}$. The ablation zone, A_{abl} , is the region below the ELA. Melt is linearly proportional to the temperature, which, in turn, decreases linearly as the tongue of the glacier recedes up the linear slope, $\tan\phi$, and increases as the glacier advances down slope. The height H of the glacier, and the width of the ablation area, w , remain constant. From Roe and O’Neal (2008).

Equation (1) is continuous in time. Roe and O’Neal (2008) show that the model discretizes naturally into time steps of $\Delta t = 1$ yr to give

$$L'_{t+\Delta t} = \gamma L'_t + \alpha T'_t + \beta P'_t, \quad (2)$$

where

$$\begin{aligned} \gamma &= 1 - \frac{\Delta t}{\tau}, \\ \alpha &= -\frac{\mu A_{T>0} \Delta t}{wH}, \\ \beta &= \frac{A_{tot} \Delta t}{wH}. \end{aligned} \quad (3)$$

τ is the dynamic response time of the glacier, or equivalently the e-folding time scale over which the glacier retains information about its previous positions. It can be thought of as the memory, the persistence, or the predictability time scale of the glacier. It is discussed more below. Note from (2) that γ is also the autocorrelation of L' at lag Δt .

2.1 Glacier characteristics.

It is useful here to review some of the basic properties of the linear glacier model, developed in Roe and O'Neal (2008). To provide a feel for the typical glacier characteristics, Table 1 is reproduced from Roe and O'Neal (2008). The largest uncertainty in glacier parameters is for the melt factor, μ , and the melt and ablation areas, $A_{T>0}$ and A_{abl} , which each have uncertainties of approximately a factor of 2.

Glacier memory. The response time, or memory, of the glacier is given by

$$\tau = \frac{wH}{\mu\Gamma\tan\phi A_{abl}}. \quad (4)$$

τ can be interpreted as the time to ablate an anomalous volume of ice: wHL' is the anomalous volume associated with an anomalous length increase, L' ; $\Gamma\tan\phi L'$ is the anomalous temperature associated with the length increase (and so $\mu\Gamma\tan\phi L'$ is the anomalous ablation rate); lastly A_{abl} is the area over which that anomalous ablation occurs. Thus, a volume in the numerator and the product of an area times a melt rate in the denominator gives a time scale. For Mt. Baker glaciers, τ is between 8 and 24 years (Roe and O'Neal, 2008).

Relative importance of precipitation and temperature. The relative sensitivity of the glacier to accumulation and ablation can be measured by the ratio of the characteristic length variations due

to melt-season temperature and accumulation alone:

$$R = \frac{\sigma_{LT}}{\sigma_{LP}} = \frac{A_{T>0}}{A_{tot}} \cdot \frac{\mu\sigma_T}{\sigma_P}, \quad (5)$$

where σ_{\cdot} is the standard deviation of variable (\cdot). R is the product of two simple ratios. The first is the ratio of the area over which ablation occurs to the area which accumulation occurs. The second is the ratio of the variabilities of melt-season temperature and accumulation, with μ being a factor that makes the ratio dimensionless. For the maritime climate of the Washington Cascades, the large accumulation variability and the muted melt-season temperature variability suggest Mt. Baker glacier lengths are more sensitive to accumulation by a factor of between two and four (Roe and O’Neal, 2008).

Natural variability of glacier length. Roe and O’Neal (2008) shows that the melt-season temperatures and annual accumulations from weather stations around Mt. Baker were neither autocorrelated nor correlated with each other. Roe and O’Neal (2008) show that, under these conditions, the standard deviation of glacier length, calculated from (2), is

$$\sigma_L = \sqrt{\frac{\tau\Delta t}{2}} \left\{ \left(\frac{A_{T>0}}{wH} \right)^2 \mu^2 \sigma_T^2 + \left(\frac{A_{tot}}{wH} \right)^2 \sigma_P^2 \right\}^{\frac{1}{2}}, \quad (6)$$

which can also be written as

$$\sigma_L = \sqrt{\frac{\alpha^2 \sigma_T^2 + \beta^2 \sigma_P^2}{1 - \gamma^2}}. \quad (7)$$

Glacier variability increases with τ – the longer the glacier memory, the longer and larger the excursions of glacier length are. The separate effects of melt season temperature and accumulation on glacier length are mediated through different aspects of the glacier geometry. For Mt. Baker glaciers and climate, Roe and O’Neal (2008) suggest σ_L is between 300 and 600 m (see Table 1).

Huybers and Roe (2008) extend (7) to derive an expression for σ_L in the case of correlated and autocorrelated climate forcing.

3 Likelihood of glacier excursions

One way of characterizing the expected natural variability of a glacier in a constant climate is to ask questions like: how long, on average, does the glacier persist above or below its equilibrium length? Or what is the expected return time, on average, of a particular glacier advance?

Such questions can be answered using standard formulas for threshold crossings of stochastic processes, first laid out by Rice (1948). VanMarcke (1983) and Leadbetter et al. (1983) contain good summaries. In the Appendix it is shown that, provided $\tau \gg \Delta t$, the average interval between up-crossings of a particular threshold, L_0 , is given by

$$R(L_0) \equiv \lambda(L_0)^{-1} = 2\pi \sqrt{\frac{\tau \Delta t}{2}} e^{\frac{1}{2} \left(\frac{L_0}{\sigma_L}\right)^2}. \quad (8)$$

This is the same thing as the average return time of glacier advances of size L_0 . For up-crossings across zero, the average return time is

$$R(0) = 2\pi \sqrt{\tau \delta t / 2}. \quad (9)$$

Therefore for the typical Mt. Baker glacier with a τ between 8 and 24 years, this means that up-crossings across zero should occur every 12 to 22 years on average. Of course much longer excursions are also possible.

Figure 2 shows average return times calculated from (8), and spanning the range of reasonable τ s for Mt. Baker glaciers. Note the logarithmic scale. The quadratic exponent in (8) means the average return time lengthens extremely rapidly as the size of the advance increases. For $\tau = 12$ yr, an advance of 1 km will happen on average every 250 yr, while for an advance of 1.5 km, the average return time balloons to 7500 yr.

It is also clear that variations in the glacier response time have an enormous impact on the average

return time. Given the complexities of real glaciers, and the imperfect information about their past variations, it is likely a practical impossibility to narrow the range of uncertainty in glacier response time to the degree that the return time for large advances can be known to within, for example, even a century.

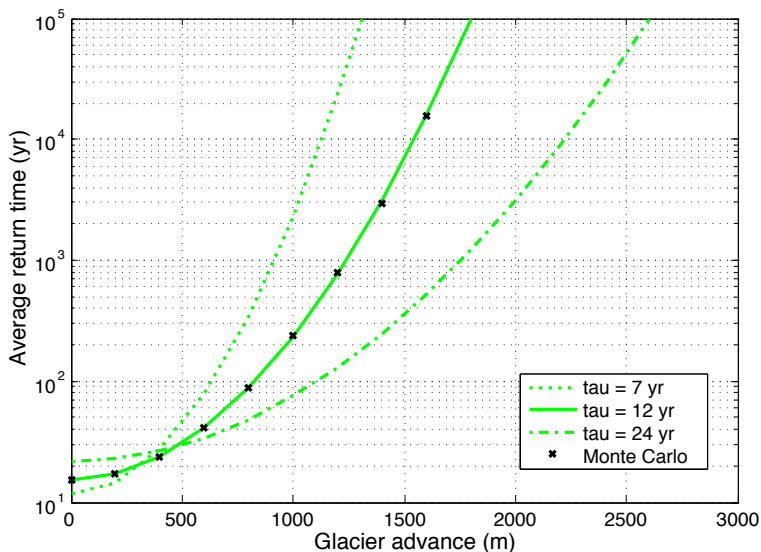


Figure 2: The average return time of a glacier advance (i.e., the interval between up-crossings of glacier length beyond a given threshold), calculated from (8). The three curves are for the range of parameters appropriate for a typical glacier on Mt. Baker, Cascades, WA. Also shown are the results of a Monte Carlo test from a 10^6 year integration of (2) with $\tau = 12$ yr. Note the logarithmic scale and the acute sensitivity of the average return time to changes in glacier properties.

3.1 Maximum glacier excursions

(8) governs the average return time of a particular advance or retreat. It is also possible to calculate the probability distribution of such return times, and this is governed by the statistics of a Poisson

distribution (e.g., Zwiers and vonStorch, 1999). A Poisson process is one in which discrete stochastic events which occur at a known rate. A requirement of the process is that a time interval, $(t_f - t_i)$, can be identified in which the likelihood of one event occurring is proportional to $(t_f - t_i)$, and that the likelihood of two events occurring in that interval is negligible.

So assuming a Poisson process, the probability of observing zero advances (or retreats) of magnitude L_0 in an interval $(t_f - t_i)$ is given by

$$p(N(t_f - t_i) = 0) = \exp[-\lambda(L_0)(t_f - t_i)]. \quad (10)$$

The probability of *at least* one occurrence of an L_0 advance (or retreat) is given by the complement of Equation (10)

$$p(N(t_f - t_i) \geq 1) = 1 - \exp[-(t_f - t_i)\lambda(L_0)] = 1 - \exp \left[-\frac{(t_f - t_i)}{2\pi} \cdot \left(\frac{2}{\tau\delta t} \right)^{\frac{1}{2}} \cdot e^{-\frac{1}{2}\left(\frac{L_0}{\sigma_L}\right)^2} \right]. \quad (11)$$

Equation (11) reveals the dependencies clearly. The probability of seeing an advance or retreat is more sensitive to $(t_f - t_i)$ than to τ . As with the return time, there is also a sensitive exponential weighting of the size of L_0 relative to σ_L . [more might be possible here](#)

The advances or retreats considered so far have been relative to the equilibrium glacier position. In any given glacial valley, it is hard to know what the long-term average position of a glacier has been, especially in the face of a changing climate. A measure of more practical relevance is the total excursion of the glacier accounting for length changes of both signs (i.e., maximum advance minus maximum retreat). The probability of a total excursion of at least a size of ΔL occurring in a given period of time is given by the probability *density* of a maximum advance between L_1 and $L_1 + dL$, $f(L_1)$, multiplied by the probability the maximum retreat exceeding $L_2 \equiv L_1 - \Delta L$, integrated over all possible maximum advances (see Figure 3). The probability

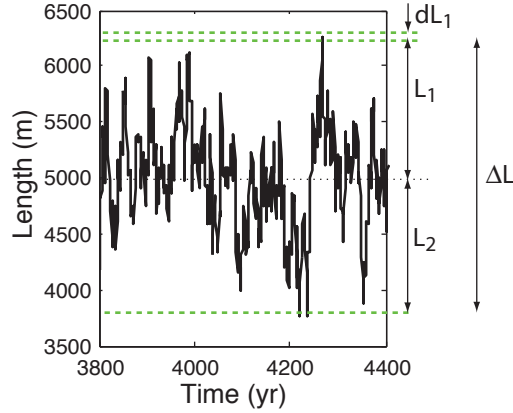


Figure 3: Schematic illustration for the calculation of the likelihood of exceeding a given total excursion.

density of a maximum advance between L_1 and $L_1 + dL$, $f(L_1)$ is,

$$f(L_1) = \frac{d}{dL_1} (N(t_f - t_i) = 0) = \frac{\lambda(L_1)\Delta L}{\sigma_L^2} e^{-(t_f - t_i)\lambda(L_1)}. \quad (12)$$

Hence the total probability of a maximum excursion exceeding ΔL in

$$p(L_{max} - L_{min} > \Delta L) = \int_0^\infty \frac{\lambda(L_1)\Delta L}{\sigma_L^2} e^{-T\lambda(L_1)} \left(1 - e^{-T\lambda(L_1 - \Delta L)}\right) dL_1. \quad (13)$$

Figure 4 reinforces the results of Reichert et al., (2001, for the Alps) and Roe and O’Neal (2008, for the Cascades). In any 100 yr period it is very likely to find an excursion of 1 km, but very unlikely to see an excursion of 3 km. As the interval under consideration increases, so do the probabilities. In any 500 yr period, there is a 50 % chance of seeing excursion longer than 2.5 km. In other words, glacier excursions of the magnitude seen at Mt. Baker (and even the Alps), often attributed to a global little ice age, should be expected to happen quite often just in the ordinary course of events. An excursion of several km, by itself, is not an indication of a climate change. As the interval being considered increases, the probabilities begin to saturate. The difference in probabilities between 100 and 500 yr is much larger than the difference in probabilities between 5000 and 10,000 yr, and the slope of the curves in Figure 4 steepens for longer intervals.

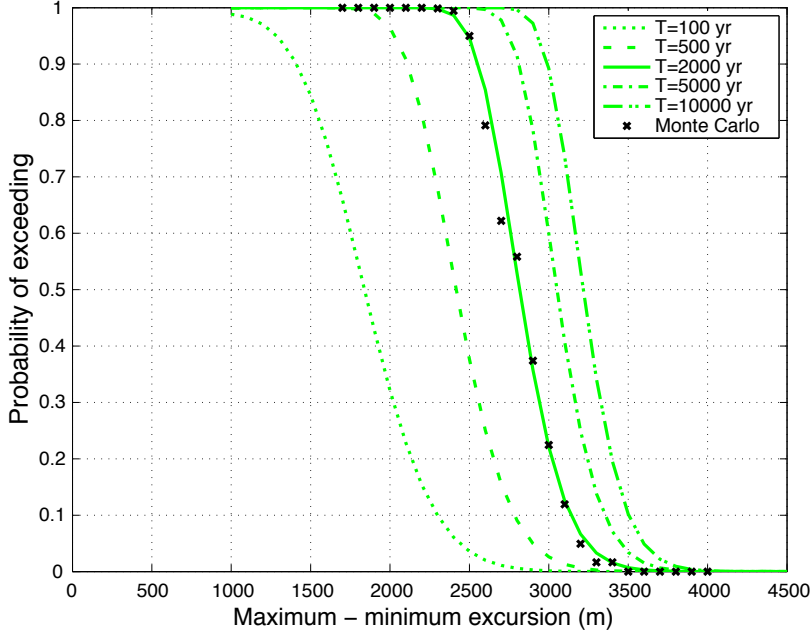


Figure 4: The probability of exceeding a given maximum total excursion (i.e., maximum advance minus maximum retreat), as a function of the interval considered, $(t_f - t_i)$, calculated from (13). Also shown for $(t_f - t_i) = 2000$ yrs is the result of a 10,000 member Monte Carlo evaluation of (13), calculated from integrations of (2).

Reichert et al. (2001) construct a similar measure to that shown in Figure 4 by collating statistics from the output of a nonlinear dynamic flowline glacier model, and choosing a criterion for a significant excursion. One advantage of using the linear model here is that the dependencies on the glacier properties are clear. Another is that definition of what constitutes a glacier excursion arises naturally from the system equations, and does not require the introduction of an arbitrary criterion.

Sensitivity to glacier properties. Just like the results for the average return time, the probability distribution of seeing large excursions is extremely sensitive to glacier properties. It depends on the tail of the probability distribution of glacier lengths, which is strongly impacted by small changes in the geometry of the glacier or in the climatic forcing. As noted above, this is due to the quadratic

exponent in (8) and (11). Figure 5 shows that the even for the relatively narrow range of glacier properties appropriate to Mt. Baker, there are very large differences in the probability of finding a given total excursion. An indication of this sensitivity is that, for the glaciers of Mt. Baker, the distance by the glaciers have retreated from their nineteenth century maxima ranges from 1.5 to 2.5 km.

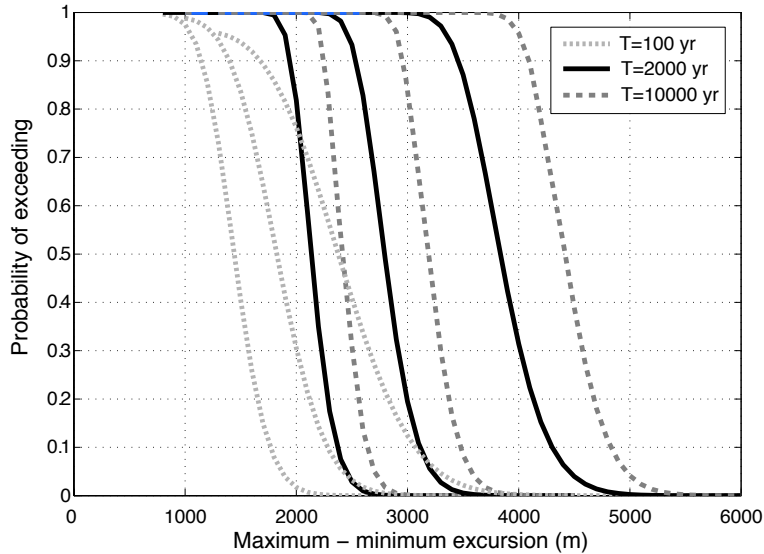


Figure 5: As for Figure 4, but each group of three curves represent the minimum, middle, and maximum set of glacier parameters appropriate to Mt. Baker (Table 1). A relatively small range of glacier properties causes an enormous range in the probability of seeing a particular glacier excursion in a given interval of time.

The two main lessons from the calculations in this section are that it is certain that large fluctuations in glacier length occur on centennial and millennial time scales, and that the magnitude of those excursions is highly sensitive to small variations in glacier properties.

4 The detection and attribution of trends in glacier length

Suppose we had not invented thermometers, but had nonetheless acquired an excellent record of how the position of a glacier terminus had varied over time. Such time series will invariably show that the glacier length has changed between the beginning and end of the period of observation. Under what conditions might we conclude that the glacier change is actually reflecting a trend in climate, rather than simply the natural fluctuations of length expected in a constant climate? This is a classic case of linear trend detection. We show here that two factors are of primary importance: the magnitude of the trend relative to the natural variability, and the amount of independent information in the observations (i.e., the degrees of freedom, in turn dependent on the length of observations and the glacier memory). The linear model can be used to understand the relative importance of these factors.

Let ρ be the correlation of L at lag Δt between the observations of glacier length and time. Standard text books on statistics (e.g., Zwiers and vonStorch, 1999) show that the following statistic follows a Student's t distribution.

$$\tilde{t} = \frac{\rho\sqrt{\nu-2}}{(1-\rho^2)}. \quad (14)$$

where ν is the degrees of freedom in the dependent time series, in this case the time series of glacier length. Standard tables can be used to evaluate whether the \tilde{t} statistic falsifies the null hypothesis that there is no trend. In general, the larger the value of \tilde{t} the greater the confidence that the observed trend is significant.

Equation (14) can also be written as:

$$\tilde{t} = \frac{b\sigma_t\sqrt{\nu-2}}{\sigma_{res}}, \quad (15)$$

where b is the regression coefficient between time and glacier length, and σ_t and σ_{res} are, respec-

tively, the standard deviations of time and of the residuals of glacier length after the time-correlated trend has been subtracted. σ_t is the standard deviation of the independent variables (in this case time) over a given interval of time, $(t_f - t_i)$, and is given by

$$\sigma_t = \left\{ \frac{1}{(t_f - t_i)} \int_{-(t_f - t_i)/2}^{(t_f - t_i)/2} t^2 dt \right\}^{\frac{1}{2}} = \frac{(t_f - t_i)}{2\sqrt{3}}. \quad (16)$$

Therefore (17) becomes

$$\tilde{t} = \frac{\Delta L}{\sigma_L} \cdot \sqrt{\frac{\nu - 2}{12}}. \quad (17)$$

$\Delta L \equiv b(t_f - t_i)$, and is the change in glacier length that is attributable to the linear trend. (17) shows some basic and readily understood dependencies.

The first term on the right hand side of (17) can be regarded as the signal-to-noise ratio: the greater the trend relative to the natural variability, the more significant the trend will be. Glaciers that exist in maritime climates are subject to a high degree of precipitation variability (e.g., Huybers and Roe, 2008), and have a muted sensitivity to temperature. As such a warming trend in melt-season temperature may be obscured by the natural variability. Continental glaciers with less precipitation variability and a higher sensitivity to temperature, therefore, are a cleaner physical system that will more directly reflect warming trends. [could give example here.](#)

The second term shows that the degrees of freedom (i.e., the number of independent pieces of information in the glacier record) is critical to the assigning statistical confidence to an observed trend. These degrees of freedom are discussed next.

Autocorrelation and degrees of freedom. If the glacier position were recorded annually there would be $N = \Delta t / \delta t$ observations. However since a glacier has a dynamical response time, and therefore memory of its previous positions, it is autocorrelated and there are fewer degrees of freedom in the time series than N . Bretherton et al. (1999) suggest that the appropriate formula for the effective

degrees of freedom is

$$\nu = N \frac{1 - \gamma^2}{1 + \gamma^2}, \quad (18)$$

where, as noted above, γ is the autocorrelation coefficient at a lag time of δt . Using the approximation that $\delta t \ll \tau$, then using (3) and taking only first order terms:

$$\nu \approx \frac{\Delta}{\delta t} \frac{\delta t / \tau}{1 - \delta t / \tau} \approx \frac{\Delta}{\tau}. \quad (19)$$

It is important to note, therefore, that the degrees of freedom do not depend on having annual observations. For a 100 yr interval and $7 \text{ yr} < \tau < 24 \text{ yr}$, (19) suggests there are between 4 and 14 degrees of freedom. **I can test the Bretherton dof formula at some point**

4.1 Is the observed trend significant?

(17) provides a way of calculating, for a single glacier, how large a change in glacier needs to be observed before the trend can be declared statistically significant:

$$\Delta L = \tilde{t}_{p=0.95, \nu} \cdot \sigma_L \sqrt{\frac{12}{\nu - 2}}. \quad (20)$$

For a 100 yr observing period, ν varies between 4 and 12, and for a 95% significance level, \tilde{t} varies between 1.76 and 2.10. For the range of σ_L that applies to Mt Baker (308 m to 630 m), glacier changes of between 540 m and 3.1 km would be necessary to declare a statistically-significant trend. The observed linear trend over the last eighty years is 146 m per 100 yr, taken from linearly detrending the compilation of results in O’Neal, 2005 for Easton, Demming, Boulder, Rainbow, and Coleman glaciers. If a shorter period, the last 30 yr, is considered, the observed trend is larger (399 m per 30 yr), consistent with an anthropogenic climate signal emerging only since that time. However the degrees of freedoms in the observations are reduced by a factor of three, and so larger glacier changes are required for statistical significance: at least 1.4 km per 30 yr. At the high end

of the model parameter range there are fewer than two degrees of freedom in a 30 year time series, and so one would conclude the record is too short to infer anything about trends..

Taking the magnitude of the observed trends into (17) and using the range of model parameters, a 146 m change over 100 yr translates to a p-value of between 0.67 to 0.53. In other words, one would expect that between 33% and 47% of random 100-yr realizations would have a trend as large as that observed. A 399 m change over 30 yr has a p-value of 0.7 at the high end of the parameter range, but is undefined at the low end.

Thus we conclude that the observed changes in Mt. Baker glaciers, by themselves, cannot be said to reflect a statistically-significant trend.

4.2 More than one glacier

More than one glacier is a good idea, but they must be independent....

5 Sensitivity to melt-season temperature vs. sensitivity to accumulation-season snowpack

Need Justin's help for this

What aspects of climate do glaciers care about? The linear model suggests simple expressions for the equilibrium changes in glacier length caused by separate changes in melt season temperature

and accumulation:

$$\begin{aligned}\Delta L_T &= -\alpha\Delta T, \\ \Delta L_P &= \beta\Delta P.\end{aligned}\tag{21}$$

From Table 1, α ranges from 49 to 88 m °C⁻¹, and $\beta = 160$ m °C⁻¹. The magnitude of the observed climate change over the twentieth century in the Cascades is somewhat uncertain due to a lack of measurements, but for the Pacific Northwest Mote et al. (2006) suggest an annual-mean temperature increase of 0.5 to 1.5 °C, and an annual-mean precipitation increase of 10% but which is highly variable spatially.

For glacier mass balance, snowpack is a key climate variable. Recent research has highlighted fundamental reasons why mountain snowpack can be extremely sensitive to changes in wintertime temperature (e.g., Casola et al., 2008; Minder et al., 2008a,b). A warmer climate raises the average altitude of the freezing level (roughly the level separating rain from snow) during a precipitation event. Except at points that are always above the freezing line, precipitation comes more often in the form of rain, reducing the snow pack there (a smaller effect is the additional melting this rain can cause). In other words, in a warmer climate less of the landscape is draped in snow, and the average depth of that snow is less. This is a robust geometric effect and more than overcomes the increase of precipitation due to the increase in moisture content of a warmer atmosphere (Minder et al., 2008a). Minder et al. (2008b) demonstrates that, in general, the sensitivity of snowpack to temperature changes depends on the hypsometry of the basin in question, and on its altitude relative to the freezing line. Specifically applying these results to the Cascades, Casola et al. (2008) find a sensitivity of mountain snowpack of 20% decrease per °C of warming. **And for Mt. Baker in particular, Minder et al. (2008a) find a sensitivity of snowpack of **% per °C.**

What changes in glacier length would be expected from these trends in temperature and snow-

pack? (21) predicts that a 0.5 to 1.5 °C increase in melt-season temperature causes a decrease in equilibrium length of between 25 and 132 m.

In the linear model, all mass balance changes act immediately to change the length of the glaciers. Thus P can be thought of as the average snowpack depth. Taking a mean snowpack of 5 m (*ref*), and a snowpack sensitivity of 20% per °C, a 0.5 to 1.5 °C increase in accumulation-season temperature, (21) predicts a decrease in equilibrium length of between 80 and 240 m. Note that the observed change over the last century lies within the range predicted by the linear model.

These results suggest that decreases in accumulation season snowpack (albeit temperature mediated) are more critical than increases in melt-season temperature for driving glacier retreats. The high sensitivity of snowpack to temperature applies generally to glaciers which sit in basins at altitudes near the freezing line. The strong difference in seasonal sensitivity also highlights the fact that temperature changes reconstructed from glacier length changes (e.g., Oerlemans, 2005) are likely to have a strong seasonal bias.

6 Summary and discussion

Obviously the global temperature is increasing, and obviously anthropogenic emissions of greenhouse gases are playing a role in that warming. Equally obviously, glaciers respond to climate changes. A harder and more subtle challenge is to ascribe climatic causes to the reconstructed past, the observed current, and the predicted future variations of glacier lengths.

The inference of past climate change from reconstructed glacial histories can only be meaningfully done in the context of the natural variability that is intrinsic to a constant climate. As a practical

matter, it is difficult to meet this standard from field observations, given the limited information typically available about glacier history, the uncertainty in the glacier dynamics, and the uncertainty in the magnitude of the stochastic climate forcing. However figure 5 does suggest that it is possible to rule out natural variability as the cause of large glacier changes even for the high range of glacier parameters. The model calculations can serve as a guide for how large such changes have to be, before they can be ascribed to climate change. Obviously also, other climate proxies can be brought to bear, but those proxies in turn must be evaluated in the context of their natural variability.

Statistical arguments regarding the detection of modern climate change from glacier length changes (e.g., Reichert et al., 2001) are contingent on modeling the glacier response, and must be treated very cautiously. At minimum a wide range of model parameters needs to be considered, and any conclusions should be regarded as fragile and tentative at best. This is not an artifact of the math formulae presented here, but rather it is fundamental to the nature of stochastically-driven systems.

- glaciers are in fact, quite messy thermometers. Their recording of temperature is corrupted by the influence of precipitation variability.

Appendix A: Threshold crossing rates

Let \dot{L} refer to the rate of change of the glacier. Rice (1948) (also Vanmarcke, 1983) showed that, for a general random process the expected rate, $\langle \lambda(L_0) \rangle$, at which a glacier advance would cross up over a given threshold, L_0 , is given by

$$\langle \lambda(L_0) \rangle = \frac{1}{2} \int_{-\infty}^{\infty} \text{sgn}(\dot{L}) p(\dot{L}, L_0) d\dot{L}. \quad (\text{A-1})$$

The term inside the integrand is the joint probability density of the glacier having a length between L_0 and $L_0 + dL$ and, simultaneously, a rate of change which would cause it to cross L_0 . The total probability is the integral over all possible rates of change. The primes have been dropped from the L s for the sake of convenience. If \dot{L} and L can be considered independent of each other, Rice further showed that the expected rate of up-crossings past L_0 is

$$\langle \lambda(L_0) \rangle = \frac{1}{2\pi} \frac{\sigma_{\dot{L}}}{\sigma_L} e^{-\frac{1}{2} \left(\frac{L_0}{\sigma_L} \right)^2}. \quad (\text{A-2})$$

where σ_L and $\sigma_{\dot{L}}$ are the standard deviations of the glacier length and its rate-of-change, respectively. We next derive an expression for $\sigma_{\dot{L}}$, and show that the correlation between \dot{L} and L can indeed be considered small.

From (1)

$$\langle \dot{L}^2 \rangle = \langle L^2 \rangle + \frac{\alpha^2 \sigma_T^2}{\Delta t^2} + \frac{\beta^2 \sigma_P^2}{\Delta t^2}, \quad (\text{A-3})$$

which, using (7), becomes

$$\sigma_{\dot{L}}^2 = \sigma_L^2 \left(\frac{1}{\tau^2} + \frac{1 - \gamma^2}{\Delta t^2} \right). \quad (\text{A-4})$$

Since we are dealing with typical conditions where $\tau \gg \delta t$ this simplifies to

$$\sigma_{\dot{L}} = \sigma_L \left(\frac{2}{\tau \Delta t} \right)^{\frac{1}{2}}. \quad (\text{A-5})$$

Next, we determine correlation coefficient between \dot{L} and L , which is given by

$$r_{\dot{L},L} = \frac{\langle \dot{L} \cdot L \rangle}{\sigma_{\dot{L}} \sigma_L}. \quad (\text{A-6})$$

From (1) it follows directly that $\langle \dot{L} \cdot L \rangle = \langle L^2 \rangle / \tau$. Therefore, using (A-5) the correlation coefficient becomes

$$r_{\dot{L},L} = \left(\frac{\Delta t}{2\tau} \right)^{\frac{1}{2}}. \quad (\text{A-7})$$

For the typical Mt. Baker parameters, $\tau \approx 12$ yr giving $r_{\dot{L},L} \approx 0.2$. Using a Monte Carlo test (Figure 2), we show that this correlation is indeed small enough to be neglected, and that therefore (A-2) provides an accurate description of threshold crossings. Substituting (A-5) into (A-2) gives

$$\langle \lambda(L_0) \rangle = \frac{1}{2\pi} \left(\frac{2}{\tau \delta t} \right)^{\frac{1}{2}} e^{-\frac{1}{2} \left(\frac{L_0}{\sigma_L} \right)^2}. \quad (\text{A-8})$$