ESS 516

Equations of motion

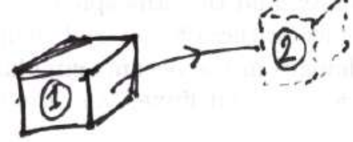
Total and Partial Derivatives

How do properties of an air parcel change as it moves around?

Imagine some scalar property of the atmosphere, ψ (e.g. temperature, pressure, etc)

In general $\psi = \psi(x,y,z,t)$ i.e. it is a function of time and space

Consider:



$$\psi_2 - \psi_1 \simeq \frac{\partial \psi}{\partial t}(t_2 - t_1) + \frac{\partial \psi}{\partial x}(x_2 - x_1) + \frac{\partial \psi}{\partial y}(y_2 - y_1) + \frac{\partial \psi}{\partial x}(z_2 - z_1)$$

~~in small limit Δ~~ n.b. 1^{st} order Taylor expansion

In small limit: $\psi_2 - \psi_1 \simeq \Delta\psi$ etc

$$\Rightarrow \frac{\Delta \psi}{\Delta t} = \frac{\partial \psi}{\partial t} + \frac{\Delta x}{\Delta t}\frac{\partial \psi}{\partial x}, \frac{\Delta y}{\Delta t}\frac{\partial \psi}{\partial y} + \frac{\Delta z}{\Delta t}\frac{\partial \psi}{\partial z}$$

using $\frac{\Delta x}{\Delta t} \sim \frac{dx}{dt} = u$

$$\Rightarrow \quad \frac{D\psi}{Dt} = \frac{d\psi}{dt} + u\frac{d\psi}{dx} + v\frac{d\psi}{dy} + w\frac{d\psi}{dz}$$

$$\boxed{\Rightarrow \quad \frac{D\psi}{Dt} = \frac{d\psi}{dt} + \underline{u}.\nabla\frac{d\psi}{dx}}$$

Rate of change FOLLOWING PARCEL (Lagrangian)

Rate of change FIXED x,y,z (Eulerian)

Advective terms

~~New~~

Equations of motion cont^d

Newtons 2^{nd} Law

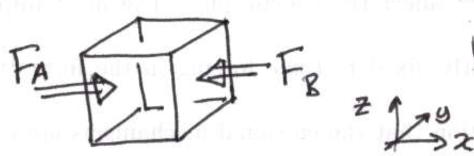
$$\underline{a} = \underline{F}/m \Rightarrow acc^n = \text{force / unit mass}$$

n.b. $\underline{a}$, $\underline{u}$ are vectors

So what are the forces

1, Gravity: $-g\underline{k}$ - points downwards!

2, Pressure gradient force



n.b. pressure is a force which acts normal to the face of the parcel

$$F_A = p_A \cdot \delta A = p_A \cdot \delta y \cdot \delta z$$

$$F_B = p_B \cdot \delta A = \left(p_A + \frac{dp}{dx}\delta x\right)\delta y \delta z$$

$$\Rightarrow \frac{F_A - F_B}{m} = -\frac{dp}{dx} \cdot \underbrace{\frac{\delta x \delta y \delta z}{m}}_{1/\rho}$$

$\delta x \delta y \delta z = \delta V$
= volume of parcel

3, Friction

- ~~small~~ mixing, due to small scale motions (turbulence). Simple representation is that it always acts to oppose direction of motion

$$\underline{F_r} = -a\,\underline{u}$$

In Summary:

$$\boxed{\underline{a} = \frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\nabla p - g\underset{\sim}{k} - a\underline{u}}$$

acceleration — pressure gradient force — gravity — friction

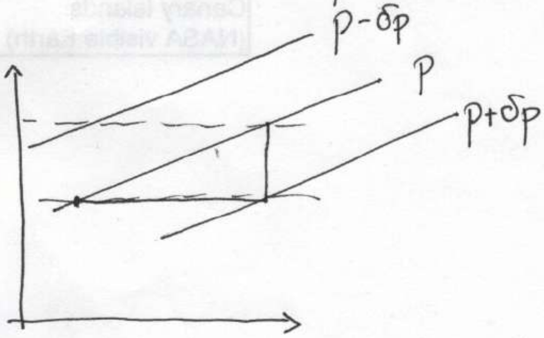
n.b. the real point of geopotential

$$\left.\frac{dp}{dx}\right)_z = -\frac{dp}{d\phi}\frac{d\phi}{dx} \qquad d\varphi = g\,dz = -\frac{dp}{\rho}$$

$$\frac{dp}{d\varphi} = -\cancel{\tfrac{1}{\rho}}\rho$$

$=$

Real Point of Geopotential



$$\Delta p = \left.\frac{dp}{dx}\right)_z \Delta x + \left.\frac{dp}{dz}\right)_x \Delta z$$

$$dp = -g\rho\, dz$$

$$\Rightarrow \cancel{\left.\frac{dp}{dx}\right)_z = - \frac{\left.dz/dx\right)_p}{dp/dz}}$$

$$\left.\frac{dz}{dx}\right)_p = \frac{-\left.\frac{dp}{dx}\right)_z}{\left.dp/dz\right)_x}$$

$$\Rightarrow \boxed{g\left.\frac{dz}{dx}\right)_p = \frac{1}{\rho}\left.\frac{dp}{dx}\right)}$$

$$\left.\frac{d\varphi}{dx}\right)_p = \frac{1}{\rho}\left.\frac{dp}{dx}\right)_z$$

$\Rightarrow$ pressure gradients on a height surface equals height gradients on a pressure surface
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