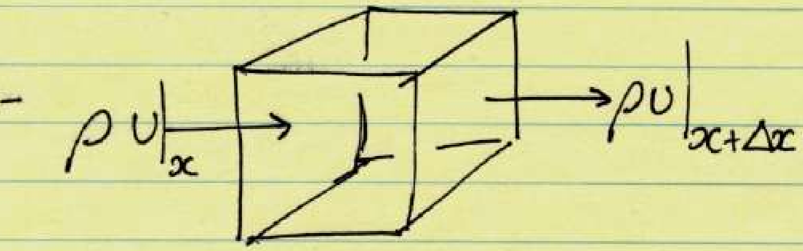
Continuity equation

- Absent any thermonuclear reactions, we have conservation of mass in the atmosphere (n.b. are there any exceptions?)

- 

- n.b. ρu is a mass flux. Units of a flux are always $\frac{\text{stuff}}{\text{unit area}} \times$ velocity.

  and a flux is a vector typically

So conservation of mass requires that

$$\frac{\partial}{\partial t} \cdot \rho \, \Delta x \Delta y \Delta z = \left(\rho u|_x - \rho u|_{x+dx}\right) \Delta y \Delta z$$

$$= -\frac{\partial}{\partial x} \rho u \quad \Delta x \Delta y \Delta z$$

or in full vector form

$$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla . \rho \underline{u} = 0$$

$$\Rightarrow \boxed{\frac{d\rho}{dt} + \rho \, \nabla \cdot \underline{u} = 0}$$

~~or in other words~~

Therefore, for an incompressible fluid: $\nabla . \underline{u} = 0$

Alternative derivation:

- following the motion, the mass contained within the 'parcel' does not change. (the parcel can, of course, get larger or smaller)

$$\Delta m = \rho \Delta x \Delta y \Delta z$$

$$= -\Delta x \Delta y \Delta p / g$$

substituting & using the hydrostatic equation

So cons. of mass

$$\frac{1}{\Delta m} \frac{d}{dt} \Delta m = 0$$

$$\Rightarrow \frac{1}{\Delta x \Delta y \Delta p} \left\{ \Delta x \Delta y \frac{d}{dt} \Delta p + \Delta p \Delta y \frac{d}{dt} \Delta x + \Delta p \Delta x \frac{d}{dt} \Delta y \right\}$$

now $\frac{d}{dt} \Delta x = \Delta u = \frac{du}{dx} \Delta x$

$\Rightarrow$ also $\frac{d}{dt} \Delta p = \Delta \omega = \frac{\partial \omega}{\partial p} \Delta p$

where $\omega = \frac{dp}{dt}$ = PRESSURE VERTICAL VELOCITY

So, whole continuity equation becomes

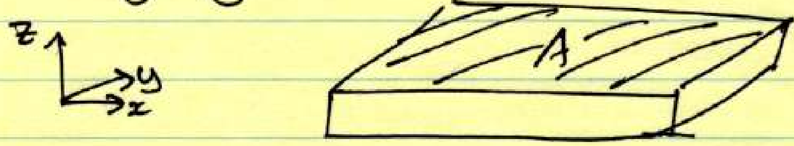
$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0}$$

- again a case of where using the pressure as a vertical coordinate is useful $\rightarrow$ we have again got rid of density from the equation

This last bit is from Wallace & Hobbs, and is great (p398)

Go back to $\frac{d}{dt}\Delta x \Delta y \Delta p = 0$

let $A = \Delta x \Delta y$, and it is the horizontal area of the 'slab' we are imagining



$A = \Delta x \Delta y$

can write

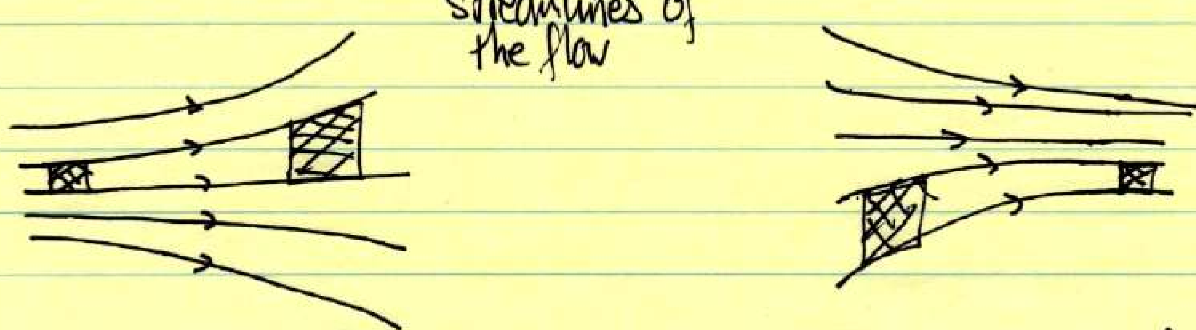
$$\Delta p \frac{d}{dt}A + A\frac{d}{dt}\Delta p = 0$$

or $$\frac{1}{A}\frac{dA}{dt} + \frac{\partial \omega}{\partial p} = 0$$

or $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{A}\frac{dA}{dt}$$

n.b. this is total derivative, so is following the 'slab': Lagrangian perspective

So can look at horizontal streamlines & have a clear idea of what is going on...



A is increasing following the parcel ($\Rightarrow dA/dt > 0$)
$\Rightarrow$ HORIZONTAL DIVERGENCE
and so $\partial\omega/\partial p < 0$

A is decreasing following the parcel ($dA/dt < 0$)
HORIZONTAL CONVERGENCE
and so $\partial\omega/\partial p > 0$