Houghton - Ch9
Holton - Ch5

Boundary layers (brief)

For a more comprehensive treatment see Holton, Hartmann, Houghton.

- Much of boundary layer theory relies on being able to separate the flow into a slowly varying component, such as the day-long changes in winds associated with the approach of a storm, and a rapidly varying component, - the minute by minute fluctuations reflecting turbulent eddies on small scales.

- There is some evidence for this from observations of wind speeds (see handouts), but note and bear in mind that a clear separation of timescales is not always evident.

- ▶ The goals is to head towards a description of what effect the fast fluctuations have on the evolution of the slow component.

But, with that in mind

$$U = \bar{U} + U'$$

$\bar{U}$ = slowly varying component = average over some time

U' = high frequency fluctuations about that average

n.b $\overline{U'} = 0$ by defn

Take momentum & continuity:

① $$\frac{du}{dt} - fv + \frac{1}{\rho}\frac{dp}{dx} = 0$$

② $$\frac{d\rho}{dt} + \nabla.(\rho \underline{u}) = 0$$

(continuity)

$$\left[\text{nb. } \frac{d\rho}{dt} + \rho \nabla.\underline{u} = \frac{\partial\rho}{\partial t} + \underline{u}.\nabla\rho + \rho\nabla.\underline{u} = \partial\rho/\partial t + \nabla.\rho\underline{u} = 0 \right]$$

iv $\rho \times (1) + u \times (2)$

first $\rho \times (1)$

$$= \rho \frac{du}{dt} + \rho u \frac{du}{dx} + \rho v \frac{du}{dy} + \rho w \frac{du}{dz} - f\rho v \pm \frac{dp}{dx} = 0 \quad (3)$$

$u \times (2)$

$$= u\frac{d\rho}{dt} + u\frac{d}{dx}(\rho u) + u\frac{d}{dy}(\rho v) + u\frac{d}{dz}(\rho w) = 0 \quad (4)$$

$(3)+(4)$

$$= \frac{d}{dt}(\rho u) + \frac{d}{dx}(\rho u^2) + \frac{d}{dy}(\rho u v) + \frac{d}{dz}(\rho u w) - f\rho v + \frac{dp}{dx} = 0 \quad (5)$$

Now, here comes the point. substitute $u = \bar{u} + u'$, $v = \bar{v} + v'$ into (5), & take time average over same interval as $\bar{u}$. The point is to derive what the net effect of the highly rapidly varying fluctuations are on the equation which describes the slowly varying component, $\bar{u}$.

n.b. $\overline{(\,)}$ means take the average of $(\,)$

DO THIS FIRST

$$\overline{(\bar{u} + u')} = \bar{u}$$

$$\overline{(\bar{u} + u')(\bar{w} + w')} = \overline{\bar{u}\bar{w} + u'\bar{w} + \bar{u}w' + u'w'}$$

these go to zero

$$= \bar{u}\bar{w} + \overline{u'w'}$$

this does not nec. go to zero

example
sine curves each average to zero
but the product does not

Okay, so
subst $u = \bar{u} + u'$ & $w = \bar{w} + w'$ etc into (5) &
then take the average

Note: density fluctuations are very much smaller than wind fluctuations $\rho = \bar{\rho}$

ives:

$$\rho \frac{d\bar{u}}{dt} + \rho\bar{u}\frac{d\bar{u}}{dx} + \rho\bar{v}\frac{d\bar{u}}{dy} + \rho\bar{w}\frac{d\bar{u}}{dz} - f\rho\bar{v} + \frac{dp}{dx}$$

$$+ \frac{d}{dx}(\rho\overline{u'u'}) + \frac{d}{dy}(\rho\overline{u'v'}) + \frac{d}{dz}(\rho\overline{u'w'}) = 0$$

OR

$$\underbrace{\frac{d\bar{u}}{dt} - f\bar{v} + \frac{1}{\rho}\frac{dp}{dx}}_{\text{evolution of slow component}} = \underbrace{-\frac{1}{\rho}\frac{d}{dx}(\rho\overline{u'u'}) + \frac{1}{\rho}\frac{d}{dy}(\rho\overline{u'v'}) + \frac{1}{\rho}\frac{d}{dz}(\rho\overline{u'w'})}_{\text{average effect of fast fluctuations}}$$

A very useful simplifying assumption to make now is that the turbulence is horizontally homogenous. That is $d/dx, d/dy \to 0$, and so the whole thing boils down to:

$$\boxed{\frac{d\bar{u}}{dt} - f\bar{v} + \frac{1}{\rho}\frac{dp}{dx} = -\frac{1}{\rho}\frac{d}{dz}(\rho\overline{u'w'})}$$

(can also derive under conditions of const density i.e. Holton)

think under what conditions this assumption will apply.

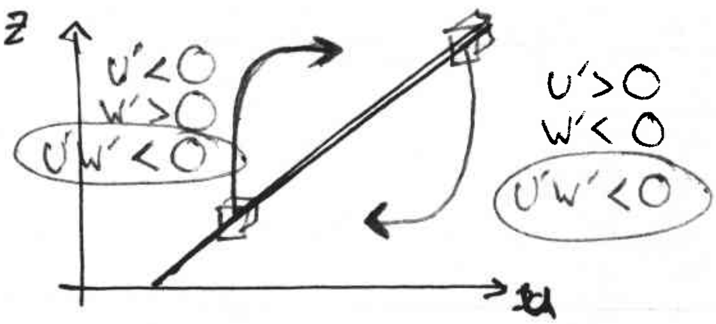
So what is this thing $\frac{1}{\rho}\frac{d}{dz}(\rho\overline{u'w'})$?

First $\rho\overline{u'w'}$ is the VERTICAL FLUX OF X-MOMENTUM

$$\left[\text{a flux} = \frac{\text{stuff}}{\text{unit area}}\cdot\frac{1}{\text{time}} = \frac{\text{stuff}}{\text{unit volume}} \times \text{speed.}\right] \quad \underbrace{\rho u}_{\frac{x\text{ momentum}}{\text{unit volume}}} \times \underbrace{w}_{\text{speed.}}$$

__Example__

wind profile



$$\boxed{\text{net effect} \quad \overline{u'w'} < 0}$$

- a turbulent eddy mixes low x-momentum upwards, and high x-momentum downwards, and so the result is a NET DOWNWARD FLUX OF MOMENTUM

- Also, in the same way $c_p\overline{w'T'}$ is the net vertical flux of heat.

So $\frac{d}{dz}(\rho\overline{u'w'})$ is the __divergence of eddy momentum flux__

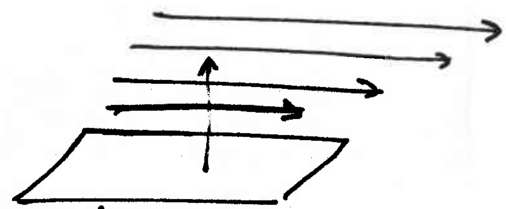
if +ve, the effect of eddies is to extract momentum from the slow component.
-ve deposit momentum into the slow component.

Secondly:

$$\rho u'w' = \frac{\text{momentum}}{\text{area time}} = \frac{\text{rate of momentum change}}{\text{area}}$$

$$= \frac{\text{force}}{\text{area}}$$

$$= \text{stress}$$



the force acts // to the surface, so it is a stress.

Hence $\rho\overline{u'w'}$ ≡ Reynold's Stresses ≡ stress acting to brake the flow.

a.k.a. "eddy stresses"

Application (1) : Well mixed boundary layer

If vigorous mixing is happening, then observations show a layer near the surface where wind speed & potential temperature are almost constant with height.

And that $\overline{u'w'} \sim C_D |V| \bar{u}$ $\qquad |V| = (u^2+v^2)^{1/2}$ = total magnitude of wind

could be linked with HW#5 from 2006

How does this compare with the Ekman spiral calculation

C_D : non dimensional drag coefficient.
$\sim 1.5\times10^{-3}$ over ocean, several times larger over land
~~4x larger~~

This is called BULK AERODYNAMIC FORMULATION & can be applied to turbulent heat fluxes too: $c_p\overline{T'w'} \sim C_D|\bar{V}|\bar{T}$. So for a boundary layer depth of h, the equation within the layer is:

$$\frac{d\bar{u}}{dt} - f\bar{v} + \frac{1}{\rho}\frac{dp}{dx} = -\left[\frac{C_D|\underline{V}|}{h}\right]\bar{u}$$

Akin to Linear friction

where $\bar{u}$ represents the depth averaged slowly varying component of the flow.

C_D = DRAG COEFFICIENT = f(atmospheric stability, wind shear)
= f(Richardson number, if you've heard of it)

Application 2 : more stable boundary layer

- by analogy with molecular diffusion, assume that momentum flux is proportional to the gradient.

$$\Rightarrow \tau = \rho \overline{u'w'} = \rho K \frac{d\bar{u}}{dz}$$

flux of momentum proportional to vertical gradient of momentum.

- sometimes called EDDY DIFFUSION, FLUX GRADIENT THEORY.

K molecular $10^{-5}\ m^2 s^{-1}$
K eddy $1 \rightarrow 10^2\ m^2 s^{-1}$

imply viscosity of treacle...
EDDIES ARE EFFECTIVE IN TRANSFERING MOMENTUM

Application 3 : mixing length hypothesis

The above assumes K is constant which is not necessarily so.

- This assumes that there is a characteristic length scale for the eddies, & that the eddy fluctuations can be described in terms of this length scale and the mean gradients in the flow.

$$u' \sim w' \sim l \frac{d\bar{u}}{dz}$$

l = length scale of eddies.

Hence $\tau = -\rho \overline{u'w'} \sim \rho l^2 \left.\frac{du}{dz}\right)^2$

shearing stress ~constant

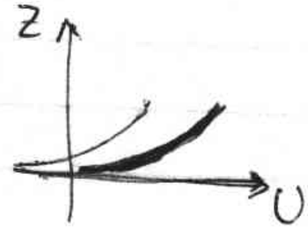
Within nearest 10m of the surface, the flux is constant

& let $l = \kappa z$ scale of eddies equals height above surface...

$$\tau = \rho \kappa^2 z^2 \left(\frac{d\bar{U}}{dz}\right)^2$$

has sol^n

$$\bar{U} = \frac{U^*}{\kappa} \ln z/z_0$$



$U^* = (\tau/\rho)^{1/2}$ friction velocity

$z_0 =$ roughness length.

κ = Von Karman constant ~ 0.4

z_0 - getting down to individual sfc. roughness elements.
- different regime.