Spr 06

# turbulence derivation

① $$\frac{du}{dt} - fv + \frac{1}{\rho}\frac{dp}{dx} = 0$$

② $$\frac{d\rho}{dt} + \nabla.(\rho \underline{u}) = 0$$

$\rho \times$①: $$\rho\frac{du}{dt} + \rho u\frac{du}{dx} + \rho v\frac{du}{dy} + \rho w\frac{du}{dz} \overset{-f\rho v}{+} \frac{dp}{dx} = 0$$

$u \times$②: $$u\frac{d\rho}{dt} + u\frac{d}{dx}(\rho u) + u\frac{d}{dy}(\rho v) + u\frac{d}{dz}(\rho w) = 0$$

$$\Rightarrow \frac{d}{dt}(\rho u) + \frac{d}{dx}\left(\frac{\rho u^2}{2}\right) + \frac{d}{dy}(\rho uv) + \frac{d}{dz}(\rho uw) - f\rho v + \frac{dp}{dx} = 0 \quad ③$$

$u = \bar{u} + u'$, $v = \bar{v} + v'$, $w = \bar{w} + w'$

& ASSUME $\rho'/\bar{\rho}, p'/\bar{p} \ll u'/\bar{u}, v'/\bar{v}, w'/\bar{w}$ (why?)

$\Rightarrow \rho = \bar{\rho} \quad p = \bar{p}$ (assume constant)

take $\overline{③}$:

$$\rho\frac{d\bar{u}}{dt} + \rho\bar{u}\frac{d\bar{u}}{dx} + \rho\bar{v}\frac{d\bar{u}}{dy} + \rho\bar{w}\frac{d\bar{u}}{dz} - f\rho\bar{v} + \frac{dp}{dx}$$
$$= -\frac{d}{dx}(\rho\overline{u'^2}) - \frac{d}{dy}(\rho\overline{u'v'}) - \frac{d}{dz}(\rho\overline{u'w'})$$

① lhs = evolution of the slowly varying components

② rhs = effect of turbulent eddies

③ assume average turbulence HOMOGENEOUS in $x, y$ dir's

$\Rightarrow \frac{d}{dx}, \frac{d}{dy} \to 0$

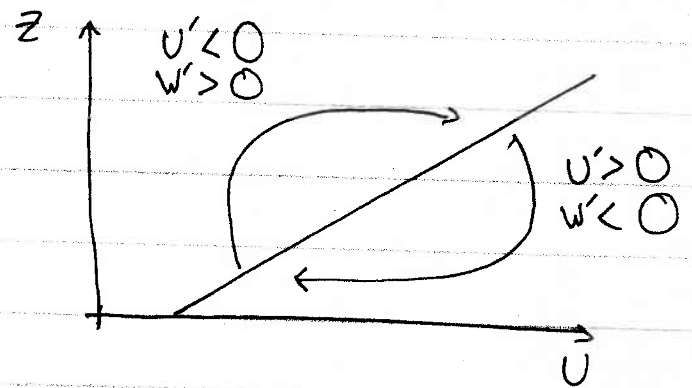
$$\Rightarrow \frac{d\bar{u}}{dt} - f\bar{v} + \frac{1}{\rho}\frac{dp}{dx} = -\frac{1}{\rho}\frac{d}{dz}\left(\rho\overline{u'w'}\right)$$

what is $\rho\overline{u'w'}$

$$\text{flux} = \frac{\text{stuff}}{\text{area} \times \text{time}} = \frac{\text{stuff}}{\text{volume}} \times \text{speed in a} \cdot$$

$$\rho u' = \frac{x\text{-momentum}}{\text{volume}}$$

$\Rightarrow \rho\overline{u'w'}$ = vertical flux of horizontal momentum.




net $\overline{u'w'}$ is negative

in this case a turbulent eddy results in a net DOWNWD FLUX OF MOMENTUM $\Rightarrow$ A DRAG

So $\frac{d}{dz}\left(\rho\overline{u'w'}\right)$ is DIVERGENCE OF VERTICAL MOMENTUM FLUX

+ve acts to decelerate mean flow

-ve acts to accelerate mean flow CONVERGENCE (PUMPING MOMENTUM IN)

# Circulation & rotation

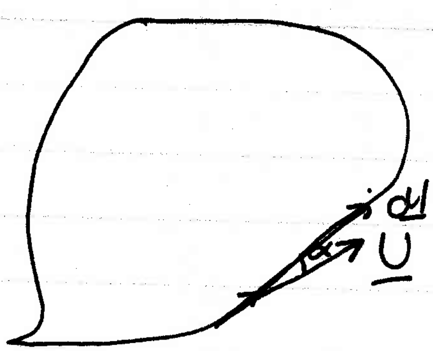


- we live on a rotating planet, and clearly rotation plays an important role in the circulations we see. We need to think about how to treat it in the atmosphere.

- The <u>circulation</u> is a measure of rotation in a fluid

 defined as the line integral evaluated along the contour of the component of the velocity that is locally tangent to the contour

$$C = \oint \underset{\sim}{U} \cdot \underset{\sim}{dl} \quad = \quad |U| \cos\alpha \, dl$$




- to show the circulation is related to the rotation, think of solid body rotation $\underline{V} = \underline{\Omega} \times R$ $\quad (v = r\omega)$

So $\cancel{C = \oint \underline{\Omega} \times}$

Let integral be a circular path around the center of rotation $C$.

ie $dl = R \cdot d\lambda$

$$\Rightarrow C = \oint \underset{\sim}{U} \cdot \underset{\sim}{dl} = \int_0^{2\pi} \underset{\sim}{\Omega} \times \underset{\sim}{R} \cdot \underbrace{R \cdot d\lambda}_{\text{arc element}}$$

$$= 2\pi R^2 \Omega$$

$$\Rightarrow C/A = 2\Omega$$

→ circulation divided by the area = TWICE the angular rotation rate.

## CIRCULATION THEOREM

take Newton's second law.

$$\frac{DU}{Dt} = -\frac{1}{\rho}\nabla P - \nabla\varphi$$

where $\nabla\varphi = g$ = gravitational geopotential.

And calculate the line integral around a closed loop:

$$\oint \frac{D\underline{u}}{Dt}\cdot d\underline{l} = \oint \frac{\nabla p \cdot d\underline{l}}{\rho} - \oint \nabla\varphi \cdot d\underline{l}$$

nb. $\frac{D\underline{u}}{Dt}\cdot d\underline{l} = \frac{D}{Dt}(\underline{u}\cdot d\underline{l}) - \underline{u}_a \cdot \frac{D}{Dt} d\underline{l}$

since $\frac{D\underline{l}}{Dt} = \underline{u}$

$$\Rightarrow \frac{D\underline{u}}{Dt}\cdot d\underline{l} = \frac{D}{Dt}(\underline{u}\cdot d\underline{l}) - \underline{u}_a \cdot d\underline{u}_a$$

$$= \frac{D}{Dt}\underline{u}\cdot d\underline{l} - \frac{1}{2} d\, u_a^2$$

note $\nabla\varphi \cdot d\underline{l} = d\varphi$ & $\oint d\varphi = 0$, $\oint d u_a^2 = 0$

$\nabla p \cdot dl = dp.$

$$\Rightarrow \boxed{\oint \frac{D}{Dt}(\underline{u}\cdot d\underline{l}) = \frac{DC}{Dt} = \oint \frac{1}{\rho}\, dp.}$$

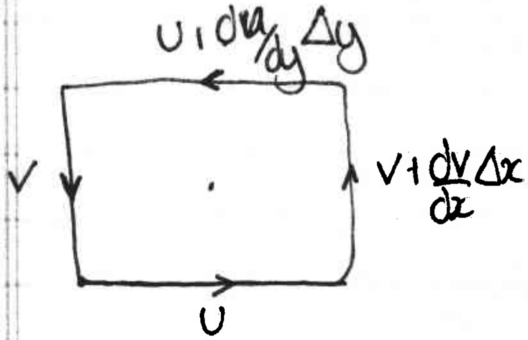
In a barotropic fluid, density is a function of pressure only

& so $\oint \frac{1}{\rho}\, dp = 0$

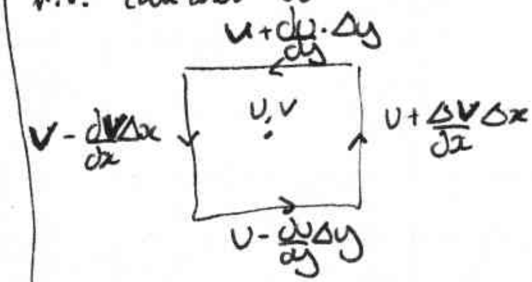
therefore $\boxed{\frac{DC}{Dt} = 0}$

CIRCULATION THEOREM ie. the circulation is conserved following the motion.

# Circulation & rotation




n.b. can also do




what is $\Delta C$?     $\Delta C$ is the circulation

$$\Delta C = u\Delta x + \left(v + \frac{dv}{dx}\Delta x\right)\Delta y - \left(u + \frac{du}{dy}\Delta y\right)\Delta x - v\Delta y$$

$$= \left(\frac{dv}{dx} - \frac{du}{dy}\right)\Delta x \Delta y$$

$$\Rightarrow \frac{\Delta C}{\Delta A} = \frac{dv}{dx} - \frac{du}{dy} = \text{ twice local rotation rate} \ldots$$

or from Stokes theorem more generally:

$$\boxed{C = \oint \underset{\sim}{u} \cdot d\underline{l} = \iint \nabla \times \underline{u} \cdot d\underline{A}}$$

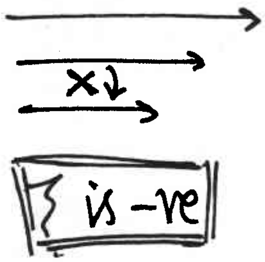
(microscopic)

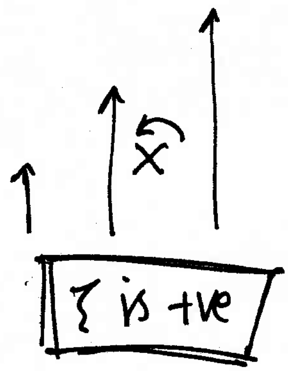
$\nabla \times \underline{u}$ is a ~~local~~ measure of rotation called VORTICITY

$C$ is a macroscopic ~~large scale~~ measure of rotation.

## "ways to get vorticity:

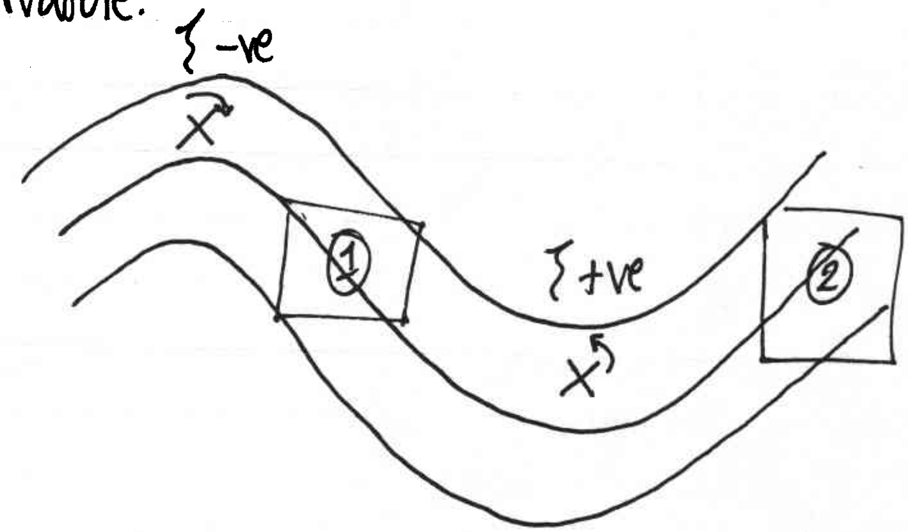
From shear:




- n.b. shear is simply a gradient in a wind field which is not aligned in the direction of the wind.

From curvature:




- n.b. in box 1 (roughly) there must be, by the vorticity equation, horizontal convergence of air.
  in box 2, there must be horizontal divergence of air.

Example:

The Earth:

$$\underset{\sim}{U}_E = \underline{\Omega} \times \underline{R} \qquad \underset{\sim}{U}_E = \text{velocity of Earth's surface.}$$

$$\nabla \times \underline{U_{surface}} = \nabla \times (\underline{\Omega} \times \underline{R})$$

now $\nabla \times (A \times B) = \underline{A}\, \nabla\cdot\underline{B} - \underline{B}(\nabla\cdot\underline{A}) - (\underline{A}\cdot\nabla)\underline{B} + (\underline{B}\cdot\nabla)\underline{A}$

since $\Omega$ const

so $\nabla \times (\underline{\Omega} \times \underline{R}) = \underline{\Omega}(\nabla\cdot\underline{R}) - \underline{R}\,\cancel{(\nabla\cdot\underline{\Omega})} - (\underline{\Omega}\cdot\nabla)\underline{R} + \cancel{(\underline{R}\cdot\nabla)\underline{\Omega}}$

$$= 3\underline{\Omega} - \underline{\Omega}$$

$$= 2\underline{\Omega}$$

$$\boxed{\begin{aligned} &R = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k} \\ &(\Omega\cdot\nabla)R \\ &= \left(\Omega_i \frac{\partial}{\partial x} + \Omega_j \frac{\partial}{\partial y} + \Omega_k \frac{\partial}{\partial z}\right)(x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}) \\ &= \underline{\Omega} \end{aligned}}$$

Moreover, the vorticity (rotation) component in the horizontal plane is

$$(\nabla \times \underline{U}_E)\cdot\underline{k} = 2\underline{\Omega}\cdot\underline{k} = 2\Omega\sin\theta = f$$

$f$ is known as vertical component of the planetary vorticity

The motion of a parcel can be thought of as having two components, one due to rotation of Earth, the other due to its relative motion

$$\underline{U}_a = \underline{U}_e + \underline{U}$$

$\underline{U}_a$ = absolute velocity
$\underline{U}_e$ = Earth sfc veloc.
$\underline{U}$ = relative velocity.

$$\nabla \times \underline{U}_a = \nabla \times \underline{U}_e + \nabla \times \underline{U}$$

ABSOLUTE VORTICITY = PLANETARY VORTICITY + RELATIVE VORTICITY

normally, only concerned with vertical components

$$\hat{k} \cdot \nabla \times \underline{U}_e = f$$
$$\hat{k} \cdot \nabla \times \underline{U} = \partial v/\partial x - \partial u/\partial y$$

Next what can be said about cons. of vorticity as the parcel moves around?

2005

$$\cancel{\frac{du}{dt} - fv +}$$

<u>Conservation of potential vorticity</u>

$$\frac{du}{dt} + \frac{1}{\rho}\frac{dp}{dx} - fv = 0$$

$$\frac{dv}{dt} + \frac{1}{\rho}\frac{dp}{dy} + fu = 0$$

$$\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz} + \frac{1}{\rho}\frac{dp}{dx} - fv = 0 \quad (1)$$

$$\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz} + \frac{1}{\rho}\frac{dp}{dy} + fu = 0 \quad (2)$$

Do $d/dy$ of (1) :

$$\left(\frac{d}{dt} + u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz}\right)\frac{du}{dy} + \frac{du}{dy}\frac{du}{dx} + \frac{dv}{dy}\frac{du}{dy} + \frac{dw}{dy}\frac{du}{dz} \quad (3)$$

$$+ \frac{1}{\rho}\frac{d^2p}{dxdy} - \frac{1}{\rho^2}\frac{d\rho}{dy}\frac{dp}{dx} - f\frac{dv}{dy} - v\frac{df}{dy}$$

Do $d/dx$ of (2):

$$\left(d/dt + u\,d/dx + v\,d/dy + w\,d/dz\right)\frac{dv}{dx} + \frac{du}{dx}\frac{dv}{dx} + \frac{dv}{dx}\frac{dv}{dy} + \frac{dw}{dx}\frac{dv}{dz} \quad (4)$$

$$+ \frac{1}{\rho}\frac{d^2p}{dxdy} - \frac{1}{\rho^2}\frac{d\rho}{dx}\frac{dp}{dy} + f\frac{du}{dx} = 0$$

∴ (3) simplifies:

$$\frac{d}{dt}\left(\frac{dv}{dx}-\frac{du}{dy}\right)+v\frac{df}{dy} = (\xi+f)\left(\frac{du}{dx}+\frac{dv}{dy}\right)$$

$$-\left(\frac{dw}{dx}\frac{dv}{dz}-\frac{dw}{dy}\frac{du}{dz}\right)+\frac{1}{\rho^2}\left(\frac{d\rho}{dx}\frac{dp}{dy}-\frac{d\rho}{dy}\frac{dp}{dx}\right)$$

$\xi$

& $$\frac{df}{dt} = v\frac{df}{dy}$$

so $$\frac{d}{dt}(\xi+f) = -(\xi+f)\left(\frac{du}{dx}+\frac{dv}{dy}\right) - \text{other terms}$$

for $u, v \sim 10\,ms^{-1}$, $w \sim 1\,cms^{-1}$ $L \sim 1000\,km$, $H \sim 10\,km$

can show that terms (5) & (6) are small

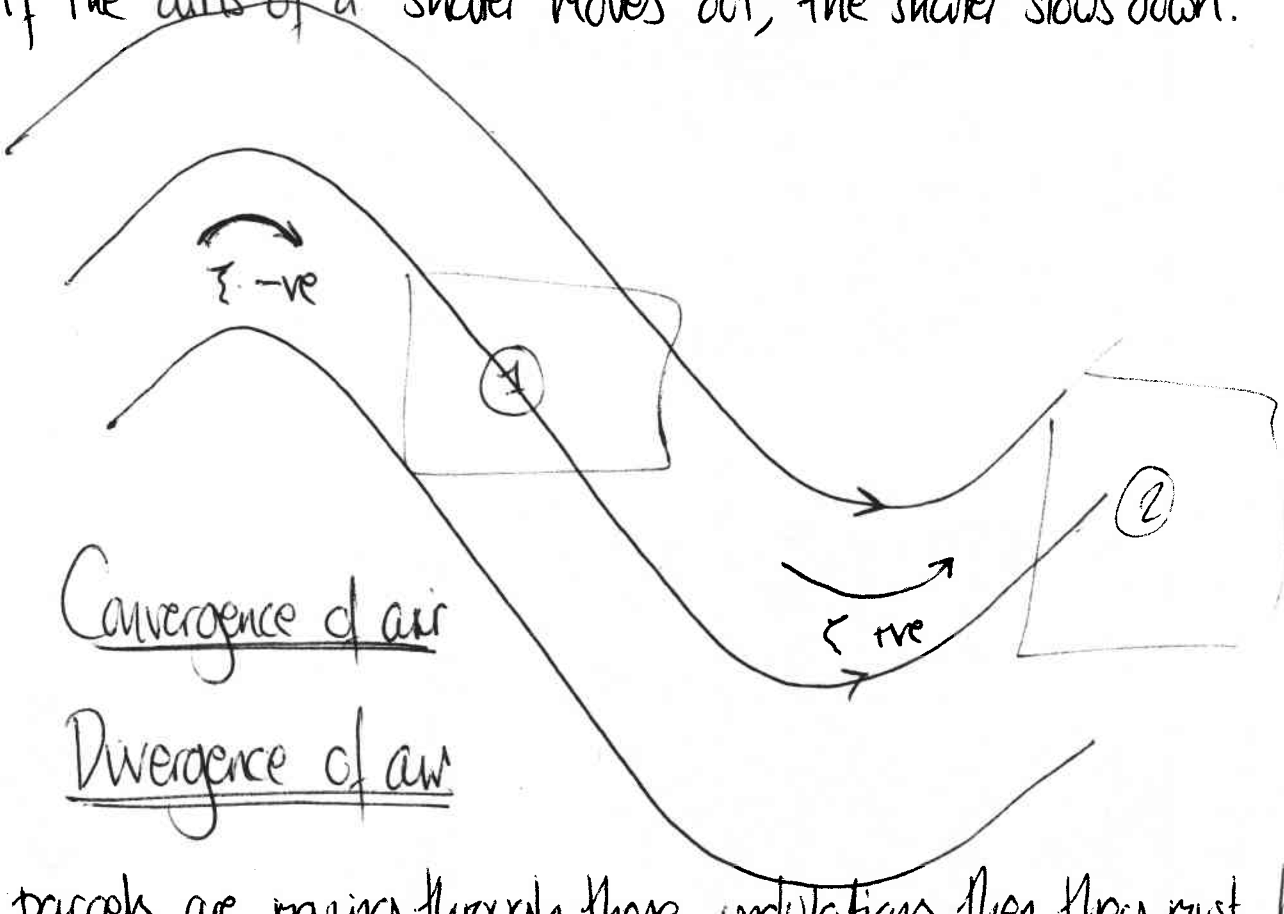
→ Also from Houghton problem 8.16, show vorticity $\xi$ may be approximated
→ by vorticity of the geostrophic wind & that $\xi \ll f$.

So for synoptic scale motions

$$\frac{d}{dt}(\zeta + f) = -(\zeta + f)\left(\frac{du}{dx} + \frac{dv}{dy}\right)$$

- if $\frac{du}{dx} + \frac{dv}{dy} > 0$ positive divergence, air flows out of a region, and vorticity decreases.

• this is analogous to conservation of angular momentum conservation, if the arms of a skater moves out, the skater slows down.




box 1 Convergence of air

" 2 Divergence of air

if parcels are moving through these undulations, then they must be experiencing convergence / divergence → ascent descent.

2008

# SCALE ANALYSIS OF VORTICITY EQ$^n$

$$\frac{d}{dt}(\xi + f) = -(\xi + f)\nabla_2 \cdot \underset{\sim}{v}$$

$$-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

① ② ③

$U \sim 10\,m/s$
$W \sim 1\,cm s^{-1}$
$L \sim 10^6\,m$
$H \sim 10^4\,m$
$\delta p \sim 10\,mb \sim 10^3\,Pa$
$\rho \sim 1\,kg m^{-3}$
$\delta\rho/\rho \sim 10^{-2}$ (same as pressure)

$f = 2\Omega \sin\theta \sim 10^{-4}\,s^{-1}$

$\tau = 1\,day\ 10^5\,s$

first $\quad \xi \sim \frac{dv}{dx} - \frac{du}{dy} \lessapprox \frac{U}{L} \sim 10^{-5}\,s^{-1}$

$$\xi/f \sim \frac{U}{fL} \equiv Ro_{\xi} \sim 10^{-1}$$

① scales as $\frac{10^{-4}}{10^5} \sim 10^{-9}\ s^{-1}$

② scales as $\frac{W}{L}\frac{U}{H} \sim \frac{10^{-2}\,10}{10^6\,10^4} \sim 10^{-11}\ s^{-1}$

③ scales as $\frac{\delta\rho\,\delta p}{\rho^2 L^2} \sim \frac{10^{-2}\,10^3}{(10^6)^2} \sim 10^{-11}\ s^{-1}$