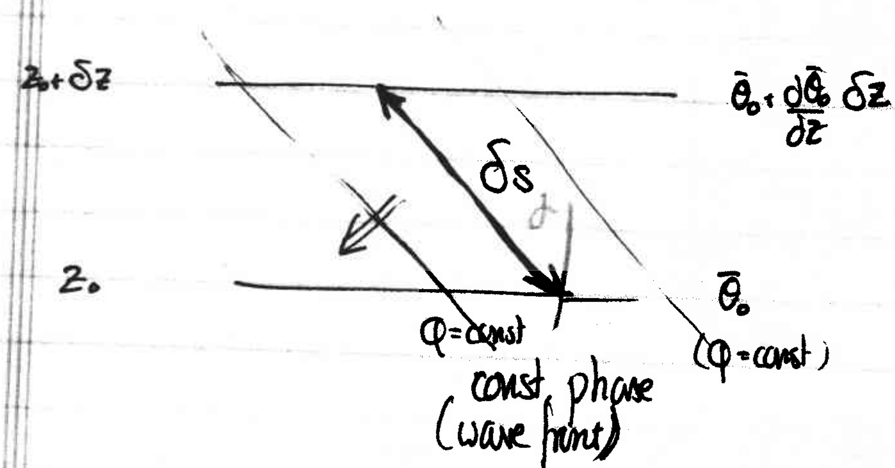
Mainly based on Holton 7.4
Houghton 8.0

Atmospheric Gravity Waves.

- Sound waves have a restoring force that is the COMPRESSIBILITY of air.
- Sound waves are LONGITUDINAL...
- Gravity waves have a restoring force that is the BUOYANCY of air in a stably stratified atmos.
- Gravity waves are TRANVERSE...



from HW#1 vertical force/unit mass $= \frac{g}{T}\left(\frac{dT}{dz} + \frac{g}{c_p}\right)\delta z$

$$= \frac{g}{\theta}\frac{d\theta}{dz}\,\delta z$$

$$= N^2\,\delta z$$

contd:

$$N^2 \delta z = N^2 \cos\alpha \, \delta s$$

Resolve forces along dirn // to wavefront:

$$\frac{d^2}{dt^2} \delta s = -N^2 (\delta s \cos\alpha) \cdot \cos\alpha$$

component of force in δs dirn

$$= -N^2 \cos^2\alpha \, \delta s$$

has soln $\delta s = e^{\pm i N \cos\alpha \, t}$

⇒ simple harmonic motion with frequency $\omega = N\cos\alpha$

depends only on the stratification & the angle of the wavefront to the vertical.

So, how to do this more formally?

① Neglect rotation

② Make Boussinesq approximation:

- density is constant except where coupled to gravity.
- atmosphere considered to be incompressible
- local variations in density small c.f. basic state
- valid where vertical motions small c.f. SCALE HEIGHT

Eq^n's of motion:

- neglect $\partial/\partial y$

x-mom eq^n $\quad \frac{du}{dt} + \frac{1}{\rho}\frac{dp}{dx} = 0$

z-mom eq^n $\quad \frac{dw}{dt} + \frac{1}{\rho}\frac{dp}{dz} + g = 0$

continuity $\quad \frac{du}{dx} + \frac{dw}{dz} = 0 \qquad$ (full eq^n $\frac{d\rho}{dt} + \rho \nabla \cdot u = 0$)

thermo $\quad \frac{d\theta'}{dt} = 0 \qquad$ & $\theta = p^{1/\gamma} \cdot \rho^{-1} \cdot$ const. from before

Again consider linearizations around a basic state

$$\rho = \rho_0 + \rho' , \quad p = \bar{p}(z) + p', \quad u = \bar{u} + u'$$
$$\theta = \bar{\theta}(z) + \theta'$$

→ Linearization is tricky, we will take 2 examples

$$\frac{1}{\rho}\frac{dp}{dz} + g = \frac{1}{\bar{\rho} + \rho'} \frac{\partial}{\partial z}(\bar{p} + p') + g$$

$$= \frac{1}{\rho}\left(1 + \frac{\rho'}{\bar{\rho}}\right)^{-1}\left(\frac{d\bar{p}}{dz} + \frac{dp'}{dz}\right) + g$$

$$\simeq \frac{1}{\bar\rho}\left(1-\frac{\rho'}{\bar\rho}\right)\left(\frac{d\bar p}{dz}+\frac{dp'}{dz}\right)+g$$ using small number expansion.

$$= \cancel{\frac{1}{\bar\rho}\frac{d\bar p}{dz}} + \cancel{g} + g\frac{\rho'}{\bar\rho} + \frac{1}{\bar\rho}\frac{dp'}{dz}$$ $\left\{\text{using } \frac{1}{\bar\rho}\frac{d\bar p}{dz} = -g\right\}$

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② Next, the eq^n of state:-

$$\theta = p^{1/\gamma}\rho^{-1}\cdot\text{const.}$$

$$\Rightarrow \ln\theta = \tfrac{1}{\gamma}\ln p - \ln\rho + \text{const.}$$

$$\Rightarrow \frac{d\theta}{\theta} = \frac{1}{\gamma}\frac{dp}{p} - \frac{d\rho}{\rho}$$

$$\Rightarrow \frac{\theta'}{\bar\theta} = \frac{1}{\gamma}\frac{p'}{\bar p} - \frac{\rho'}{\bar\rho}$$

solve for ρ': $\rho' = -\bar\rho\frac{\theta'}{\bar\theta} + \frac{p'}{c^2}$ $\qquad c^2 = \frac{\gamma\bar p}{\bar\rho}$ = spd. of sound

CHECK for buoyancy motions $\left|\bar\rho\cdot\theta'/\bar\theta\right| \gg \left|p'/c^2\right|$

- Density variations due to pressure changes <u>much</u> smaller than those due to temperature changes

So can write:

$$\frac{\theta'}{\bar{\theta}} = -\rho'/\rho_0$$

Crucial difference between Gravity waves & Sound waves See Holton.

O.K., phew:

linearizing eq^{n}s then gives:

$$\left(\partial/\partial t + \bar{U}\,\partial/\partial x\right)u' + \frac{1}{\rho_0}\frac{\partial p'}{\partial x} = 0 \quad ①$$

* linearization gives $\rho'/\bar{\rho}$ which becomes θ'/θ upon subst. of above.

$$\left(\partial/\partial t + \bar{U}\,\partial/\partial x\right)w' + \frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\theta'}{\bar{\theta}}g = 0 \quad ②$$

$$\partial u'/\partial x + \partial w'/\partial z = 0 \quad ③$$

$$\left(\partial/\partial t + \bar{U}\,\partial/\partial x\right)\theta' + w'\,\partial\bar{\theta}/\partial z = 0 \quad ④$$

Head towards a wave eq^n:

eliminates p'

$$\partial/\partial x\,② - \partial/\partial z\,① : \quad \left(\partial/\partial t + \bar{U}\,\partial/\partial x\right)\left(\frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial z}\right) - \frac{g}{\bar{\theta}}\frac{\partial \theta'}{\partial x} = 0 \quad ⑤$$

Do $\left(\partial/\partial t + \bar{U}\,\partial/\partial x\right)\partial/\partial x$ of ⑤

$$\Rightarrow \left(\partial/\partial t + \bar{U}\,\partial/\partial x\right)^2\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 u'}{\partial x \partial z}\right) + \frac{g}{\bar{\theta}}\frac{\partial^2}{\partial x^2}\left(\partial/\partial t + \bar{U}\,\partial/\partial x\right)\theta' = 0$$

& using ③ & ④

$$\boxed{\left(\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial x}\right)^2 \left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2}\right) - \underbrace{\boxed{\frac{g}{\bar{\theta}}\frac{\partial \bar{\theta}}{\partial z}}}_{= N^2} \frac{\partial^2 w'}{\partial x^2} = 0}$$

& this is a wave eqⁿ

trial solⁿ $\quad w' \sim w_0 \, e^{i(kx + mz - \omega t)}$

again $\partial/\partial t \sim -i\omega$, $\partial/\partial x \sim ik$, $\partial/\partial z \sim im$

so becomes

$$(-i\omega + \bar{U} i k)^2 (-k^2 - m^2) + N^2 k^2 = 0$$

$$\boxed{\omega = \bar{U} k \pm \frac{Nk}{\sqrt{k^2 + m^2}}}$$