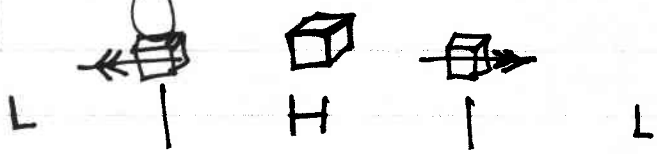
Sound waves

- Sound waves are longitudinal waves: motion is // to dirⁿ of propagation
- alternating regions of compression and rarefaction set up pressure gradients which cause oscillations of air parcels



Look a problem in 1-D (possible because of longitudinal nature of waves)

1. Neglect rotation
2. Assume motion is adiabatic

Eqⁿs of motion

$$\frac{du}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \quad (1)$$

$$\frac{d\rho}{dt} + \rho\frac{\partial u}{\partial x} = 0 \quad (2)$$

$$\frac{d\theta}{dt} = 0 \quad (3)$$

In 1-D $\partial/\partial y, \partial/\partial z \to 0$

why have we neglected friction / rotation?

Can you justify this?

Begin with thermodynamic equation:

$$\theta = T\left(\frac{p_0}{p}\right)^{R/c_p}$$

$T = p/\rho R \Rightarrow \theta = \frac{p}{\rho R}\left(\frac{p_0}{p}\right)^{R/c_p}$

head towards var^n of density & pressure

$$= p^{1-R/c_p} \cdot \rho^{-1} \cdot \text{const.}$$

$$= p^{1/\gamma} \cdot \rho^{-1} \cdot \text{const} \qquad \gamma = c_p/c_v$$

if θ is constant as parcel moves around $\left(d\theta/dt = 0\right)$, then

take logs of both sides.

$$1/\gamma \ln p - \ln \rho = \text{const.}$$

$$\boxed{\text{or } \frac{1}{\gamma}\frac{d}{dt}\ln p - \frac{d}{dt}\ln \rho = \cancel{\text{const}}\ 0}$$

This gives relationship between pressure perturbations & density perturbations following a parcel of air.

$\Rightarrow$ from 2 & thermo, therefore.

$$\frac{1}{\rho}\frac{d\rho}{dt} = \frac{d}{dt}\ln\rho = \frac{1}{\gamma}\frac{d}{dt}\ln p = \frac{1}{\gamma p}\frac{dp}{dt}$$

so 2 eq^{n}s of motion:

$$\frac{du}{dt} + \frac{1}{\rho}\frac{dp}{dx} = 0 \quad (1)$$

substitute into (2)

$$\frac{1}{\gamma p}\frac{dp}{dt} + \frac{du}{dx} = 0 \quad (2)$$

Method of solution is to assume a small perturbation to a BASIC STATE - time averaged values of u, p, ρ etc.

let basic state $\bar{u}, \bar{\rho}, \bar{p}$ = const.

then $p = \bar{p} + p', \rho = \bar{\rho} + \rho', u = \bar{u} + u'$ (3)

where $p', \rho' \ll \bar{p}, \bar{\rho}$ u' small, (not nec. $\ll \bar{u}$ - still works)

$u'\rho' \ll \bar{u}\rho'$ etc

Subst. (3) into (1) & (2), & neglect terms of order $U'p'$...

<u>Example</u>

$$\cancel{\partial/\partial t + (\bar{U}+U')\partial/\partial x}$$

(4)

$$\frac{\partial(\bar{U}+U')}{\partial t} + (\bar{U}+U')\partial/\partial x(\bar{U}+U') + \frac{1}{\bar{\rho}+\rho'}\,\partial/\partial x\,(\bar{p}+p') = 0$$

$\frac{\partial}{\partial x}, \partial/\partial t$ of $\bar{U}, \bar{p} = 0$.

and $$\frac{1}{\bar{\rho}+\rho'} = 1/\bar{\rho}\left(1+\rho'/\bar{\rho}\right)^{-1} = 1/\bar{\rho}\left(1 - \frac{\rho'}{\bar{\rho}} + \left(\frac{\rho'}{\bar{\rho}}\right)^2 + O\left(\frac{\rho'}{\bar{\rho}}\right)^3\right.$$

& higher)

(small no. expansion)

$$= 1/\bar{\rho}\left(1-\rho'/\bar{\rho}\right)$$

(4) becomes

$$\frac{\partial U'}{\partial t} + \bar{U}\frac{\partial U'}{\partial x} + U'\frac{\partial U'}{\partial x} + 1/\bar{\rho}\left(1-\rho'/\bar{\rho}\right)\partial/\partial x\;\bar{p}' = 0$$

taking only terms of order ( )'.

$$\boxed{\Rightarrow \left(\partial/\partial t + \bar{U}\frac{\partial}{\partial x}\right)U' + 1/\rho\,\frac{\partial p'}{\partial x} = 0}$$

2 eqⁿs become:

1 $$\left(\partial/\partial t + \bar{U}\,\partial/\partial x\right)u' + \frac{1}{\rho}\frac{\partial p'}{\partial x} = 0$$

2 $$\& \quad \left(\partial/\partial t + \bar{U}\,\partial/\partial x\right)p' + \gamma\bar{p}\frac{\partial u'}{\partial x} = 0$$ ⇐ HW Qⁿ Show this to be true...

$\partial/\partial x$ ①, & substitute into $(\partial/\partial t + \bar{U}\partial/\partial x)$ ②

$$\Rightarrow \boxed{\left(\partial/\partial t + \bar{U}\,\partial/\partial x\right)^2 p' - \frac{\gamma\bar{p}}{\rho}\frac{\partial^2 p'}{\partial x^2} = 0}$$

this is a wave eqⁿ, so look for solutions of the form

$$p' = p_0 e^{i(kx-\omega t)}$$ & take real part.

for this type of solⁿ $\quad \partial/\partial t \to -i\omega, \; \partial/\partial x \to ik$

$$\Rightarrow (-i\omega + \bar{U}ik)^2 + \frac{\gamma\bar{p}}{\rho}.-k^2 = 0$$

$$(\omega - \bar{U}k)^2 - \frac{\gamma\bar{p}}{\rho}k^2 = 0$$

$$\boxed{\omega = \bar{U}k \pm k.\sqrt{\frac{\gamma\bar{p}}{\rho}}}$$

Phase speed of wave: $c = \omega / k$

$$c = \bar{U} \pm \sqrt{\frac{\gamma \bar{p}}{\bar{\rho}}}$$

2 sol^{n}s. Phase speed relative to flow $\sqrt{\frac{\gamma \bar{p}}{\bar{\rho}}}$

Note that sound waves are <u>non dispersive</u>. All wavelengths have the same speed. The group velocity is a constant $c_g = c_p = d\omega/dk$.