GEOPHYSICS, VOL. 51, NO. 12 (DECEMBER 1986); P. 2247–2249, 4 FIGS.

Short Note

A stable free-surface boundary condition for two-dimensional elastic finite-difference wave simulation

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INTRODUCTION

Two of the persistent problems in finite-difference solutions of the elastic wave equation are the limited stability range of the free-surface boundary condition and the boundary condition's treatment of lateral variations in velocity and density. The centered-difference approximation presented by Alterman and Karal (1968), for example, remains stable only for β/α greater than 0.30, where β and α are the shear (S) and compressional (P) wave velocities. The one-sided approximation (Alterman and Rotenberg, 1969) and composed approximation (Ilan et al., 1975) have similar restrictions. The revised-composed approximation of Ilan and Loewenthal (1976) overcomes this restriction, but cannot handle laterally varying media properly.

First we present a free-surface boundary approximation that is stable for all physical β/α ratios and correct for laterally varying media. The method is implicit, but only requires a simple pentadiagonal system solver to implement. When the proposed boundary conditions are coupled with second-order and fourth-order approximations for the elastic wave equation, the overall problem is stable for β/α greater than 0.01 and β/α greater than 0.02, respectively. A simple numerical test of the method and a comparison with other published methods are given in the second section.

FREE-SURFACE BOUNDARY CONDITIONS

The two-dimensional free-surface boundary conditions of zero tangential and normal stress are

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 \tag{1}$$

and

$$\gamma \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{2}$$

where u and w are the horizontal and vertical displacements, x and z are the horizontal and vertical spatial coordinates, and γ is $(1 - 2\beta^2/\alpha^2)$. To apply finite differences to equations (1) and (2), an extra row is introduced above the actual free surface. The geometry is shown in Figure 1. The standard finite-difference approximations to the elastic wave equation (Kelly et al., 1976) can be used to determine the solution on the interior of the mesh, up to and including row 1. Previous free-surface boundary conditions cited above have used explicit finite-difference approximations to determine row 0. This approach can cause instability for small values of β/α and lead to problems with laterally varying media due to the difficulty in centering the normal and tangential derivatives concurrently with averaging laterally varying parameters of the media.

The method proposed here uses an implicit formulation which centers both the normal and tangential derivatives at the free surface, halfway between row 0 and row 1. The scheme is similar in concept to the Crank-Nicholson method for the diffusion equation (e.g., Claerbout, 1976, p. 185). Assuming the vertical and horizontal grid spacings are equal and applying centered second-order differences to equations (1) and (2), then

$$\mathbf{u}_0 - \tfrac{1}{4}\underset{\sim}{\mathbf{B}}\mathbf{w}_0 = \mathbf{u}_1 + \tfrac{1}{4}\underset{\sim}{\mathbf{B}}\mathbf{w}_1 \tag{3}$$

and

$$\mathbf{w}_0 - \tfrac{1}{4}\underset{\sim}{\mathbf{\Gamma}}\underset{\sim}{\mathbf{B}}\mathbf{u}_0 = \mathbf{w}_1 + \tfrac{1}{4}\underset{\sim}{\mathbf{\Gamma}}\underset{\sim}{\mathbf{B}}\mathbf{u}_1, \tag{4}$$

where the subscripts denote the solution on rows 0 and 1, $\underset{\sim}{\mathbf{\Gamma}}$ is a diagonal matrix which contains the values of γ across the surface, and $\underset{\sim}{\mathbf{B}}$ is a bidiagonal matrix with subdiagonals and superdiagonals equal to -1 and 1, respectively. Note that centering of the x derivatives is achieved by averaging estimates on rows 0 and 1.

Equations (3) and (4) can be reduced to separate systems for the unknown vectors $\mathbf{u}_0$ and $\mathbf{w}_0$:

$$(\underset{\sim}{\mathbf{I}} - \tfrac{1}{16}\underset{\sim}{\mathbf{B}}\underset{\sim}{\mathbf{\Gamma}}\underset{\sim}{\mathbf{B}})\mathbf{u}_0 = (\underset{\sim}{\mathbf{I}} + \tfrac{1}{16}\underset{\sim}{\mathbf{B}}\underset{\sim}{\mathbf{\Gamma}}\underset{\sim}{\mathbf{B}})\mathbf{u}_1 + \tfrac{1}{2}\underset{\sim}{\mathbf{B}}\mathbf{w}_1, \tag{5}$$

Manuscript received by the Editor May 20, 1985; revised manuscript received April 7, 1986.
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and

$$(\mathbf{I} - \tfrac{1}{16}\mathbf{\Gamma B}^2)\mathbf{w}_0 = (\mathbf{I} + \tfrac{1}{16}\mathbf{\Gamma B}^2)\mathbf{w}_1 + \tfrac{1}{2}\mathbf{\Gamma B u}_1. \tag{6}$$

The matrices on the left side of equations (5) and (6) are pentadiagonal and can be solved rapidly by an algorithm that is a simple extension of the standard tridiagonal solver (e.g., Claerbout, 1976, p. 189). The vectors on the right side can be computed from displacements on row 1.

To show the basic stability of equations (5) and (6) as extrapolation operators, we assume γ is constant and perform a Fourier transform over x. This step leads to the system

$$\begin{bmatrix} u_0 \\ w_0 \end{bmatrix} = \frac{1}{1 + \frac{\gamma}{4}\sin^2 kh}\begin{bmatrix} 1 - \frac{\gamma}{4}\sin^2(kh) & i \sin kh \\ i\gamma \sin kh & 1 - \frac{\gamma}{4}\sin^2(kh) \end{bmatrix}\begin{bmatrix} u_1 \\ w_1 \end{bmatrix}, \tag{7}$$

where u and w now denote the Fourier duals of $\mathbf{u}$ and $\mathbf{w}$, k is the dual of x, and h is the horizontal mesh spacing. The moduli of the eigenvalues of the matrix in equation (7) are unity, indicating that the boundary conditions do not amplify the wave field. In other words, in the "frozen coefficient" problem, the free-surface boundary conditions are stable. To show that the combined problem of the boundary conditions and the interior solution is stable is beyond the scope of this note. In numerical tests, the combined problem remained stable for β/α greater than 0.01 for a second-order interior method, and for β/α greater than 0.02 for a fourth-order method. We suspect that the stability problem lies with our interior solutions rather than with the boundary conditions.

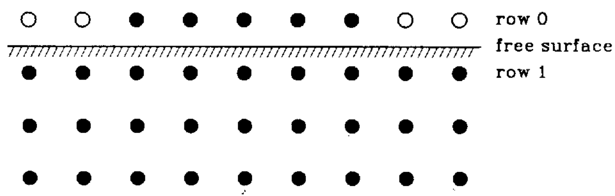

FIG. 1. Free-surface geometry. Row 0 is a fictitious row added to allow finite differences of horizontal derivatives. The true free surface lies halfway between row 0 and row 1. The open circles require special treatment in the boundary conditions.

EDGES OF THE FREE SURFACE

The boundary conditions derived above must be modified for the two extreme edge elements on each side of the free surface, shown as open circles in Figure 1. For these points, we apply the $B1$ absorbing boundary conditions of Clayton and Engquist (1977). For the component u on the left side of the grid, the boundary conditions in equation (5) are modified to be

$$(1 + \delta)u_0^t + (1 - \delta)u_1^t = (1 - \delta)u_0^{t-1} + (1 + \delta)u_1^{t-1} \tag{8}$$

and

$$(1 + \delta)u_1^t + (1 - \delta)u_2^t = (1 - \delta)u_1^{t-1} + (1 + \delta)u_2^{t-1}, \tag{9}$$

where $\delta = \alpha\Delta t/h$, h is the mesh spacing, and Δt is the time step. Here (u_0, u_1, u_2) are the first three elements of the vector $\mathbf{u}_0$, and superscripts t and $t - 1$ refer to the present and previous time steps. Stability of equations (8) and (9) is independent of the β/α ratio. These equations most effectively absorb horizontally traveling P-waves. Similar equations are used for the vertical component w, except that $\delta = (\beta\Delta t)/h$ is used to absorb horizontally traveling S-waves. Mirror images of these conditions are used at the right edge of the free surface.

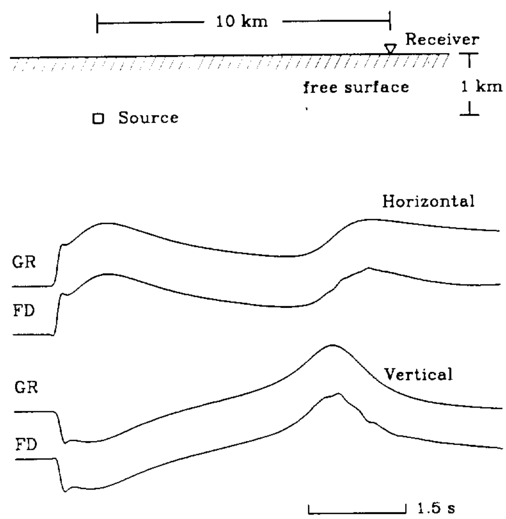

FIG. 2. The analytic result for a half-space with an explosive line source is compared to a fourth-order finite-difference (FD) solution with the implicit free-surface boundary condition. The analytic result is calculated by the generalized ray (GR) method of de Hoop (1960). In the half-space, α is 3.5 km/s and β is 1.5 km/s. For this calculation, the time step is 0.01 s and the grid spacing is 0.05 km.

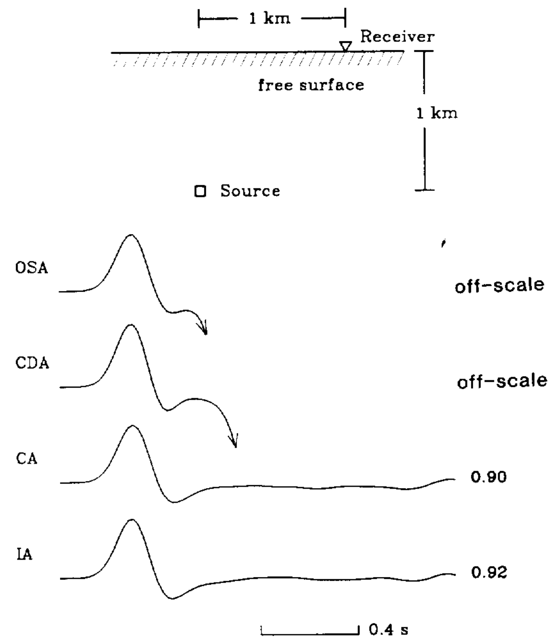

FIG. 3. The vertical component for the one-sided (OSA), central-difference (CDA), revised-composed (CA), and implicit (IA) approximations of the free-surface boundary conditions are shown for the small β/α ratio of 0.2. Traces are not shown where they go off scale. Peak amplitudes are given to the right of each trace. In the half-space, α is 3.5 km/s and β is 0.7 km/s. An explosive line source is used. For this calculation, the time step is 0.002 5 s and the grid spacing is 0.025 km.

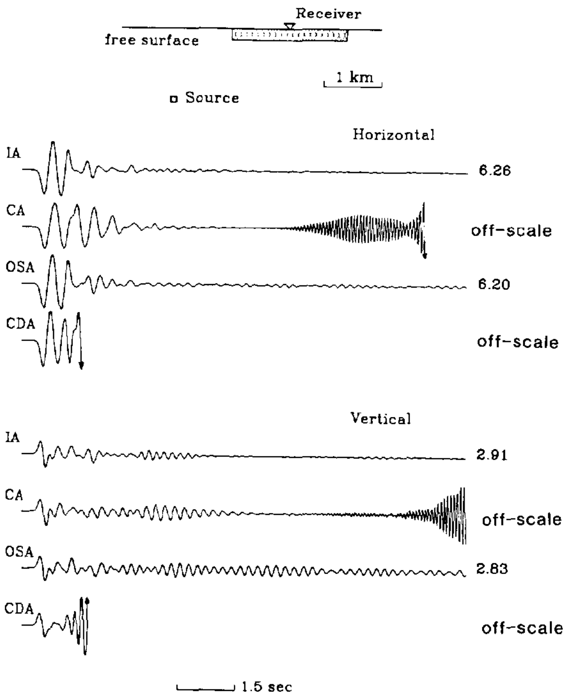

FIG. 4. The horizontal and vertical component for the one-sided (OSA), central-difference (CDA), revised-composed (CA), and implicit (IA) approximations of the free-surface boundary conditions are shown for a laterally varying structure. In the hatched region, which is 2.0 km wide and 0.1 km deep, α is 1.3 km/s, β is 0.6 km/s, and ρ is 1.0 g/cm^3. In the rest of the half-space, α is 3.5, β is 2.0 km/s, and ρ is 2.6 g/cm^3. Peak amplitudes are given to the right of each trace. An explosive line source is used. For this calculation, the time step is 0.002 5 and the grid spacing is 0.025 km.

NUMERICAL EXAMPLES OF STABILITY AND ACCURACY

The accuracy of the proposed implicit scheme is shown in Figure 2. The revised-composed scheme and central-difference scheme (not shown) are about as accurate as the implicit scheme because all three are accurate to second order. The one-sided scheme (also not shown) is only accurate to first order and therefore introduces more error at the shorter wavelengths. Both the direct wave and the Rayleigh wave agree with the analytic calculation. The slight difference in sharpness between the analytic trace and the finite-difference trace is due to grid dispersion in the finite-difference method. The small perturbations to the Rayleigh wave are S-wave noise from the source region.

The stability of the proposed free-surface boundary conditions is illustrated by calculations for the vertical component of a half-space problem where β/α is 0.2. Results are shown in Figure 3. The revised-composed scheme and implicit scheme are well-behaved, while the one-sided scheme and central-difference scheme are not. The slight difference in amplitude between the revised-composed scheme and implicit scheme results from the slight difference in the free-surface position, which is half a mesh spacing farther from the source with the revised-composed scheme than with the implicit scheme. The results for the stability of the one-sided scheme, central-difference scheme, and revised-composed scheme are in agreement with those found by Ilan and Loewenthal (1976).

A test with lateral heterogeneity is shown in Figure 4. The velocities within the rectangle under the receiver are a factor of 3 less than the velocities in the half-space. The one-sided scheme is inaccurate, particularly on the horizontal component. The central-difference scheme is unstable. The revised-composed scheme is more gently unstable; however, in our experience, structure with more lateral variation causes the method to become unstable more rapidly. Part of the disagreement between the revised-composed scheme and the implicit scheme in the early part of the record arises from the slightly different location of the free surface mentioned above. Only the implicit scheme results in energy dying away with time. We suspect the result is accurate, but we have no method for conveniently checking it.

The only free surface which is stable for both low β/α ratios and lateral heterogeneity is the implicit scheme proposed here.

ACKNOWLEDGMENTS

This work was partly supported by AFSOR contract F19628-83-K-0010, and a research grant from Ametek Inc. J.E.V. was supported by an NSF fellowship. Critical reviews by John Louie, Heidi Houston, and anonymous reviewers were helpful.

Contribution no. 4227, from the Div. of Geological and Planet. Sciences, California Institute of Technology, Pasadena, CA, 91125.

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