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Short Note

Rapid calculation of seismic amplitudes

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INTRODUCTION

The ability to calculate traveltimes and amplitudes of seismic waves is useful for many reflection seismology applications such as migration and tomography. Traditionally, ray tracing (Ĉerveny et al., 1977; Julian, 1977), paraxial methods (Claerbout, 1971), or full-wave methods (Alterman and Karal, 1968) are used for such calculations. These methods have in common considerable computational expense. Recently, Vidale (1988, 1990a) presented two-dimensional and three-dimensional methods to efficiently compute traveltimes of the first arrivals to every point in a regularly spaced grid of points, given an arbitrary velocity field sampled at these points. The computational cost of finding each traveltime is roughly one square root operation.

This note describes how the amplitudes of the first arrivals can be calculated in two dimensions without explicit knowledge of the raypaths. The traveltimes for a set of four closely spaced sources are sufficient to determine the geometric amplitudes for each point in the grid in the case of a smoothly varying velocity structure. The computation cost for each amplitude is six square roots and an arctangent. This low cost allows the computation of many more amplitudes than is possible with ray tracing or waveform methods. Alternatively, this method of amplitude estimation can be done using other less efficient but more accurate traveltime calculations, such as ray tracing or ray bending (Prothero et al., 1988; Thurber, 1983) and needs four to sixteen traveltimes per amplitude.

METHOD

The method is described in two dimensions for simplicity, although the extension to three dimensions is straightforward. Conceptually, the amplitudes are inversely related to the degree of spreading of a small cone of rays emanating from the source. The calculation has two steps: first, the takeoff angle from the source to each grid point is found; then, the variation in takeoff angle between adjacent points around a receiver is used to estimate the geometric spreading and thus the amplitude.

Once a velocity structure has been specified, the traveltimes to all points in a two-dimensional (2-D) grid are found by the method of Vidale (1988) for four sources located at s0, s1, s2, and s3 around the source point s shown in Figure 1. The grid points are assumed to be regularly spaced.

For each grid point, then, the traveltimes from the four sources are known. In laterally varying structure, the ray parameter is not invariant and can no longer be used to find the takeoff angle and its derivatives from only the horizontal derivatives of traveltime. Since, by reciprocity, the traveltime from a source to a receiver is the same as the traveltime from the receiver back to the source, the four raypaths in Figure 1 can also be considered to be from a single source to four receivers. The difference between the traveltimes for points s1 and s3 measures the apparent velocity of the raypath projected on the horizontal axis near the source, and the difference between s0 and s2 measures the vertical apparent velocity. The four traveltimes, taken together with the assumption of an incident plane wave, yield an estimate of the propagation direction of the reciprocal "first arrival" to point s. Thus the takeoff angle of the first arrival from s to the receiver is given by

$$i = \arctan\left(\frac{t3 - t1}{t0 - t2}\right), \tag{1}$$

where i is the takeoff angle from the central point s between s0, s1, s2, and s3, measured from the vertical, and tj are the traveltimes from source sj for j equals 0, 1, 2, and 3. This takeoff angle can be calculated to all points on the grid.

The concept of three-dimensional (3-D) geometric ray tubes is that at sufficiently high frequencies, seismic energy stays in a tube whose sides are all geometric raypaths (Ĉerveny et al., 1977). The 2-D analog is that seismic energy is contained by a wedge-shaped region whose boundary is two geometric rays. In Figure 2, for example, energy trans-

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mitted from the source into the region between the rays from the source to points r0 and r2 stays in that region; this is a high-frequency approximation. Since the wedge-shaped region grows wider as the energy travels farther from the source, the energy density diminishes with time; this is the phenomenon of geometric spreading. With the common assumption that the source radiation is isotropic very near the source, the width of the wedge as a function of space is the only information required to find the geometric spreading factor.

The amount of energy incident on the receiver is proportional to the angle Δi subtended at the source of the wedge (shown in Figure 3) that illuminates a line through the receiver normal to the raypath (in three dimensions this line would be a surface) with a width of twice the grid spacing. This angle Δi is equivalent to the ratio $J(0)/J(s)$ that measures geometric spreading in ray-tracing calculations [equation (39), Červeny and Hron, 1980, for example].

The angle Δi may be found from the takeoff angles to the four points surrounding the receiver shown in Figure 2 by

$$\Delta i = \sqrt{(i0 - i2)^2 + (i1 - i3)^2}, \tag{2}$$

where $i0$ is the takeoff angle of the ray from the source to the point r0, and similarly for points r1, r2, and r3. This finite-difference formula assumes that the variation in takeoff angle across the points near the receiver is a linear function, which is equivalent to assuming the amplitude is constant to the four points.

The amplitude of the acoustic wave at the receiver also depends on the velocity and density at the receiver (see Hall, 1987, for example), and thus the formula for relative pressure amplitude across the grid is

$$A_p(x, z) = \sqrt{\Delta i(x, z) u(x, z) \rho(x, z)}, \tag{3}$$

where u is acoustic wave speed and ρ is density. This formula may be used to compute the amplitude at any grid point once given the amplitude at one point. The equivalent formula for motion such as displacement, velocity, or acceleration is

$$A_m(x, z) = \sqrt{\frac{\Delta i(x, z)}{u(x, z)\rho(x, z)}}. \tag{4}$$

The equation for a cylindrical geometry would also have terms that are trigonometric functions of the takeoff angle and the angle of incidence. These formulas are most accurate for small velocity contrasts, smooth velocity variations, and high frequencies. Sharp velocity contrasts will not be properly included since equation (3) assumes no loss of energy through reflection. More exotic radiation patterns can be inserted without difficulty by scaling the angle i (see below) by the radiation pattern appropriate for the source. Sources or receivers on the surface may be buried one level down in the grid with little change in the geometric spreading.

The amplitude calculation, when used with the traveltime calculation of Vidale (1988), will only compute the amplitude of first arrivals. Amplitudes that are computed by finite-differencing across points separated by a cusp in the traveltime isocron, which is equivalent to a discontinuous jump in the raypath where the intermediate raypaths have folded

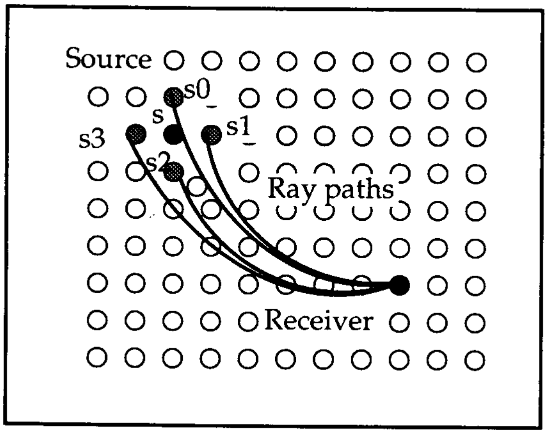

FIG. 1. Diagram of the four raypaths connecting the receiver and the four grid points s0, s1, s2, and s3 closest to the source s.

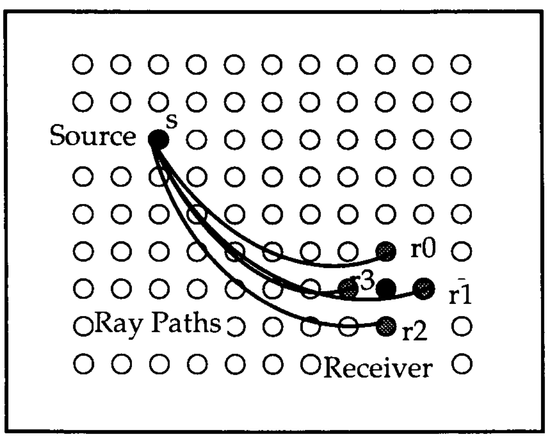

FIG. 2. Diagram of the four raypaths connecting the source s and the four points closest to the receiver.

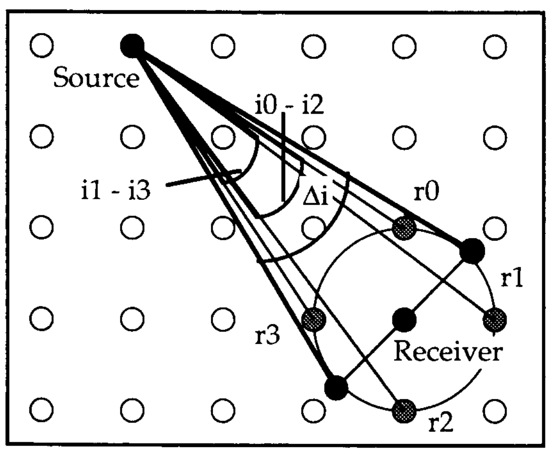

FIG. 3. The angles subtended at the source by the pair of points r0 and r2 (angle $i0$-$i2$), the points r1 and r3 (angle $i1$-$i3$), and a line with a width of two grid spaces (angle Δi).

back into secondary arrivals, will show large amplitudes. The amplitudes are due to the relatively large change in takeoff angle to points across the cusp, combined with the inability of the method of Vidale (1988) as it is currently formulated to follow secondary arrivals. Diffractions are correctly treated; each point in the neighborhood finite-differenced has a seismic ray that left the source with the same takeoff angle, so no energy is in the ray tube and the geometric amplitude of the diffraction is zero.

A NUMERICAL EXAMPLE

Here we compare our new amplitude calculation with amplitudes determined by finite-differencing of the full-wave equation. The velocity model (in Figure 4a) may be considered to resemble a cross-borehole survey. The calculation of amplitude from a surface source to a reflector at depth would have a similar geometry.

The traveltime method of Vidale (1988) has been slightly modified to compute the times to points within three grid points of the source more accurately, since high precision is necessary to compute amplitudes. Any traveltime calculation method may be used; if amplitudes at isolated points are required, 16 traveltimes are needed (four for each of the four measurements of takeoff angle). If a sufficiently dense set of amplitudes is desired, only four traveltimes need be computed for each grid point since each time may be shared between four neighboring grid points. The spherical and cylindrical equivalents of equations (1) through (4) may be derived easily. Only one representative example is presented because the errors and artifacts are predominantly due to the traveltime scheme, which is documented in Vidale (1988).

Broad but strong velocity variations are required to compare this geometric amplitude calculation with full-wave methods. The 10 percent rms velocity variations shown in Figure 4 predominantly have wavelengths of 50 grid points or longer on the 128 by 128 point grid. Density is kept constant in the test; properly accounting for velocity variations is the more demanding task. Figure 4 also shows the isochrons of traveltime. The isochrons show the expected pattern: rays passing through the slow regions are retarded and rays passing through the fast regions are advanced.

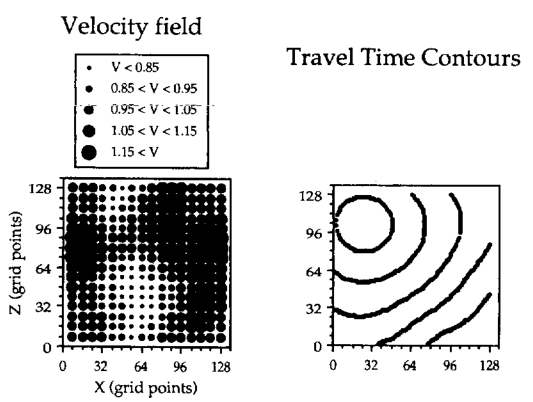

FIG. 4. (left) Map of the velocity field across the 128 by 128 point grid. (right) 25, 50, 75, 100, and 125 s contours of traveltime.

The 2-D full-wave amplitudes may be directly compared with geometric ray results. Although full-wave line sources disperse somewhat near the source region and finite frequency waves can diffract, use of the derivative of a Gaussian as the source time function minimizes dispersion and use of long-wavelength velocity variations minimizes diffraction.

The results of the new scheme are plotted next to the results of a full-wave finite-difference calculation in Figure 5. Figure 5a shows the traveltime amplitudes from a 128 by 128 point grid. The 128 by 128 grid is averaged over 8 by 8 squares (to form a 16 by 16 grid point image) since the results must be legible in publication. Only the first-arrival amplitudes are recovered. The full-wave amplitudes shown in Figure 5b are taken from the peak value of synthetic seismograms. For this comparison, the source should have a relatively short duration so that amplitude is well defined and also a relatively narrow bandwidth to limit the line-source dispersion. To meet these requirements, we chose as our time function the derivative of a Gaussian pulse (te^{-t^2}, where t is time).

The full-wave calculation is done with an acoustic fourth-order finite-difference algorithm (Vidale, 1990b) and is accurate for the frequencies and distances used in this case. The grid has been interpolated from 128 by 128 to 512 by 512 to allow simulation of higher frequencies. Incidentally, the full-wave calculation is reduced from 10 hours to 1 hour by using the traveltime calculation (Vidale, 1988) to window the area over which the full-wave calculation is active; that traveltime calculation takes 2 minutes for a 512 by 512 grid.

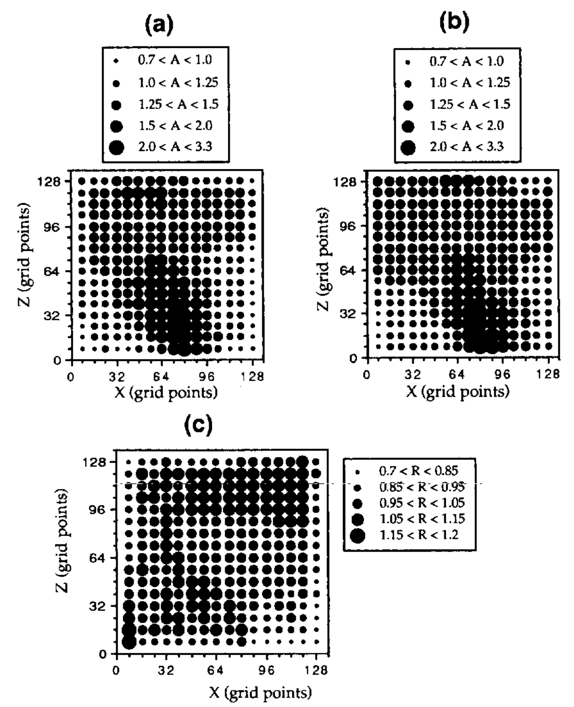

FIG. 5. (a) Amplitudes calculated for the velocity model shown in Figure 4 with our new method. Amplitudes are normalized to remove geometric spreading in a whole space. (b) Amplitudes from full-wave finite-difference calculation. (c) Ratios of the amplitudes from our new method to those from the full-wave calculation.

The ratio of the two results is plotted in Figure 5c. Note that most of the points are within 10 percent of perfect agreement, while the amplitudes vary by a factor of four. Experiments with several test cases indicate that averaging over a 2 by 2 point square is necessary to avoid fine-scale oscillations. Alternatively, the four sources shown in Figure 1 and the four receivers shown in Figure 2 could have radii of two grid points rather than one. With the larger radii, there is little reason to determine the amplitude at every grid point since some smoothing has been applied. An efficient calculation would compute the traveltimes on a fine grid, then resample the traveltimes to a grid a factor of two coarser, and finally find takeoff angle and amplitudes only for the coarse grid with source and receiver arrays of radius of one grid point. With this procedure, the set of four traveltimes on the coarse grid would require no more computer memory than each of the four traveltime calculations on the fine grid.

Good agreement is seen in Figure 5. The agreement degrades slightly where a caustic forms to the bottom right of the low-velocity zone in the middle of the grid. This is expected because the full-wave calculation propagates finite-frequency waves rather than the infinite-frequency waves assumed by ray theory. Minor artifacts in the traveltime amplitudes that appear when the method of Vidale (1988) is used to compute traveltime may be seen in the ratio. Energy propagating parallel to the rows and columns of the grid is slightly amplified and energy traveling diagonally is slightly diminished.

The accuracy of the traveltimes produced by the method of Vidale (1988) increases as the grid spacing is made finer. Stencils for the finite-difference operators that span more grid points in equations (1) and (2) produce a smoother, more numerically noise-free estimate of the amplitude. Thus for more accuracy at the cost of more computation to find the amplitudes at a given number of grid points, one would compute the traveltimes on a grid a factor of two finer, then compute the amplitudes with stencils that are a factor of two wider.

CONCLUSIONS

This new method of computing amplitudes has several advantages over existing schemes. The foremost advantage is the great efficiency: only six square roots and an arctangent give the traveltime and amplitude for each point. The error in the method is about 10 percent in several test cases against full-wave methods. Caustics and diffractions may readily be seen in amplitude maps. Knowledge of the raypaths is not necessary. A limitation of this method is that it yields geometric amplitudes. The extension to three dimensions is straightforward. This method promises to be useful for economical prestack migration, tomographic inversion, and other operations in which rapid forward modeling of seismic amplitudes would be advantageous.

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REFERENCES

Alterman, Z., and Karal, F. C., 1968, Propagation of elastic waves in layered media by finite-difference methods: Bull., Seis. Soc. Am., **58**, 367–398.

Ĉerveny, V., Molotkov, I. A., and Pŝenĉik, I., 1977, Ray methods in seismology: Univ. of Karlova Press.

Ĉerveny, V., and Hron, F., 1980, Ray synthetic seismograms for *SH* waves in anelastic media: Bull., Seis. Soc. Am., **70**, 49–78.

Claerbout, J. F., and Johnson, A. G., 1971, Extrapolation of time-dependent waveforms along their path of propagation: Geophys. J. Roy. Astr. Soc., **26**, 285–295.

Hall, D. E., 1987, Basic acoustics: Harper and Row Publ.

Julian, B. R., and Gubbins, D., 1977, Three-dimensional seismic ray tracing: J. Geophys. Res., **43**, 95–114.

Prothero, W. A., Taylor, W. J., and Eickemeyer, J. A., 1988, A fast, two-point, three-dimensional raytracing algorithm using a simple step search method: Bull., Seis. Soc. Am., **78**, 1190–1198.

Thurber, C. H., 1983, Earthquake locations and three-dimensional crustal structure in the Coyote Lake area, central California: J. Geophys. Res., **88**, 8226–8236.

Vidale, J. E., 1988, Finite-difference traveltime calculation: Bull., Seis. Soc. Am., **78**, p. 2062–2076.

———, 1990a, 3-D finite-difference traveltime calculation: Geophysics, **55**, 523–528.

———, 1990b, Comment on "A comparison of finite-difference and Fourier method calculations of synthetic seismograms" by C. R. Daudt, L. W. Braile, R. L. Nowack, and C. S. Chiang: Bull., Seis. Soc. Am., **80**, 493–495.