

1 DEFINITION OF A CONTINUUM by C.F. Raymond

What is continuum mechanics about?

Mechanics examines the motions of bodies, which are described by their size, shape, mass, moment of inertia, position of center of mass, velocity, momentum, energy, etc. and which interact through various field or contact forces. The goal is to describe the motion of the bodies relative to one another. Examples are trajectories, orbiting bodies interacting by gravity, charged particles moving in electric and magnetic fields.

Continuum mechanics comes into play when we want to examine what is going on inside a body in a smoothed picture that does not go into detail about the forces and motions of the atomic or other sub-scale particle constituents. It thus introduces an idealization (approximation) of the material of a body that is defined as follows: A continuum is a region of space filled with continuous matter with continuous properties. By continuous we mean that various properties averaged on a length and time scale of interest vary smoothly within the region except possibly for a small number of discontinuities. Since everything consists of small bits (atoms) and possibly larger heterogeneous pieces, whether this is possible depends on the scale of interest in relation to the actual structure of the material.

There are various types of continuum properties of interest, primarily related to what it is made of (constitution), how its constituents are moving (kinematics) and forces acting in it (dynamics).

We consider first constitution. Smoothly varying intrinsic physical properties of the material should be definable. Examples of commonly relevant properties are mass density, specific heat capacity, thermal conductivity, permeability to water flow, elasticity and viscosity, all of which we will define in due course.

Let us now take total mass density ρ as an example. Define

$$\bar{\rho}_r = \frac{\text{mass in } V_r}{V_r}$$

where V_r is a volume with characteristic dimension r , which is the scale of the averaging. If r is very small, say of the order of molecular dimensions, then clearly $\bar{\rho}_r$ will show dramatic variations with position, for example as in Fig. 1 (a) below. If we are considering a gas, liquid, or some mineral, then

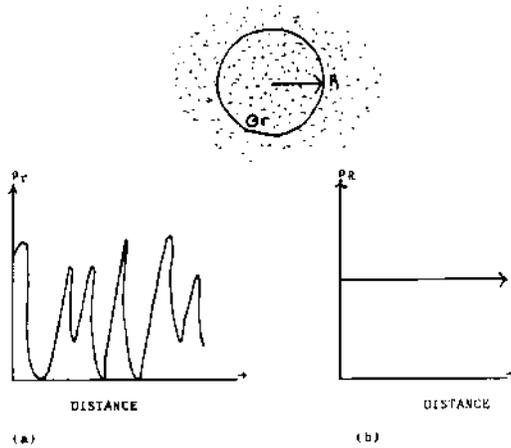


Figure 1: Density of a material averaged over dimensions that are similar to (a) and much larger (b) than the spacings of constituent molecules or atoms.

if r is much larger than molecular dimensions, $\bar{\rho}_r$ is well defined and changes only slowly with position as in Fig. 1 (b).

Another illustrative example of a constitutive property is thermal conductivity. What can be said about the thermal conductivity of a rock composed of grains of differing minerals, say a granite. Granite is an aggregate of several minerals: feldspar, quartz, biotite, and others. Each mineral grain can safely be considered a continuum on averaging scales smaller than the grain size (typically order 1 millimeter) and larger than molecular dimensions. To some extent the different mineral types have different thermal conductivity. If we were to consider heat flow with spatial resolution on the order of a single or small number of grain sizes, then the relevant thermal conductivity would show substantial local variations. On this scale we would have to deal with the properties of each constituent mineral and there is no well-defined single continuum. We would have to consider it as a complex aggregate of discrete continua and take account of all of the boundaries between them. On the other hand, if we consider an averaging scale that is much larger than the grain size, then the thermal conductivity would vary smoothly. On this scale we can think of a single continuum, granite, with its own properties which represent some sort of average of the properties of the constituent minerals.

Now we consider kinematics. In a continuum it must be possible to define local displacement and velocity that vary continuously, except possibly for

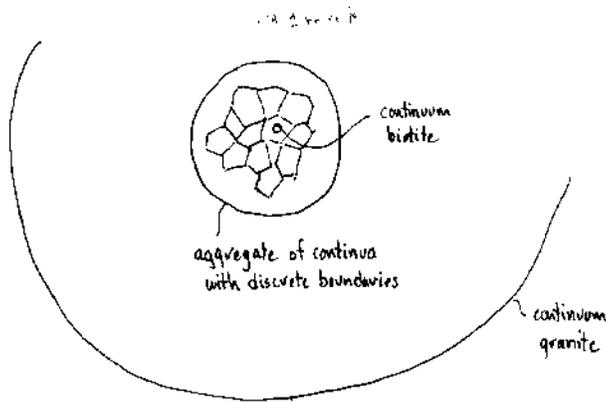


Figure 2: Continuum view of a rock at different scales.

a small number of discontinuities on internal boundaries called faults. That means that the description may be constrained to appropriate spatial and temporal averaging scales.

In a gas individual molecules have an extreme range of velocities associated with thermal motions, but with an averaging scale which includes a very large number of molecules it is possible to define a mean velocity which in our ordinary experience is wind speed. In deforming sand, the displacement and velocity of pieces of matter on the scale of individual sand grains is very discontinuous, but smoothly varying quantities can be defined with a larger scale. Consider also turbulent water or other fluid. On a small scale there may be irregular disorganized motion (turbulence). Although this small scale motion is present, principal interest may be in a larger scale view of the motion. The contributions to the motion from eddies can be averaged out; then the continuum is not regular water but turbulent water with different viscosity and diffusion properties.

With regard to forces, it should be possible to define force densities which vary continuously. There are two main types. The first type is called volume force and is the force on unit volume of material arising from some force field. In geophysics the most common volume force is the force of gravity acting on a material with a well-defined mass density. Another example is an electric field acting on a well-defined charge density. The second type is called surface force and is the force acting across unit area of any internal or external surface. The first of these is often called body force and the second

is usually called stress or sometimes traction.

If one considers some of the earlier examples, it is clear that there is also a problem of scale in defining the body force and stress. Air pressure is a surface force (stress) that arises from the time/space average of molecules impinging on a surface. Stress in deforming sand would represent the mean force per unit area arising from the local forces acting across particle contacts.

The question of whether a material (or materials) can be considered a continuum for the solution of some practical problem or analysis of some data is often a subjective matter. The concept of a continuum is an idealization, which may approximate reality, when one looks in appropriate distance and time averages. Usually in geophysics we find examples where the concept of a continuum is clearly applicable. A dense plasma, atmospheric air, ocean water, glacier ice, a large body of granite are examples. However, there are cases where the answer is not so clear. Suppose a large body of granite is cracked (has joints) every several or several tens of meters and we consider some relevant constitutive property, for example, the way water moves through it (permeability). As a whole the body will not behave the same way as a small sample of coherent granite behaves in a lab test. Possibly on an averaging scale of several hundred meters we could idealize the body as a continuum (e.g. permeability of cracked granite), but this is not always obvious.

The goal of continuum mechanics is to predict the distribution of displacements and forces within a body when prescribed forces and/or displacements are applied to its outer surface. To do this we must learn how to (1) describe the essential aspects of the displacement field, (2) describe the state of forces, and (3) have knowledge of the mechanical behavior of the material (i.e., kinematic response to certain forces). Essentially this boils down to learning how to predict the response of an arbitrary continuous body under arbitrary surface loads or displacements from the results of several simple experiments such as stretching a rod or twisting a cylinder. This is especially successful when the continuum is one on a scale of mm, cm, or even meters which can be handled in the lab. In geophysics we are often stuck with a larger scale and must rely on natural experiments.