Magnetosphere Configuration

• High latitude, open field lines, magnetospheric convection, magnetotail
• Middle altitudes: plasma sphere/ radiation belts, ring current
• Ionosphere: aurora, field aligned currents, acceleration mechanisms, field aligned currents
• Global current systems linking them all
Next: Start learning about the difference between Cold and Hot plasma motions – they are VERY different in the magnetosphere
FIG. 10.4. Schematic diagram of plasma regions of the earth's magnetosphere as viewed in the noon–midnight meridian plane. The plasmasphere typically occupies much of the same region of space as the radiation belts. Frequently there is little or no gap between the inner edge of the plasma sheet and the outer boundary of the trapped radiation belts.
FIG. 10.18. Typical patterns of (a) Birkeland current, (b) electric field, (c) horizontal ionospheric current, and (d) \( \mathbf{E} \times \mathbf{B} \)-drift velocity observed in the earth's ionosphere, as viewed from high above the North Pole. Local noon is toward the top of the page, local dusk to the left, and so forth. These patterns represent the primary ionospheric effects of magnetospheric convection. The

\[
\phi(x) = \int x \cdot ds
\]
Single Particle Motion in E and B fields

Everything follows from Lorentz Force:

1st assume time independent fields

- Gyration
- ExB drift: charge and energy independent
- Grad-B and curvature drifts: depend on both charge and energy (explains How the Radiation Belts Work)
- Adiabatic invariants of the motion in a dipole field
Single particle motion in B field

Start with Lorentz Force

\[ m \frac{d\vec{v}}{dt} = q \vec{v} \times \vec{B} \]

Simple harmonic Oscillator

\[ \omega_c = \frac{eB}{m} \]

\[ r_L = \frac{v_L}{\omega_c} = \frac{m v_L}{eB} \]

Larmor radius

Now add \( \vec{E} \) field and transform away \( \vec{E} \)

\[ \vec{F} = 0 = q \vec{E} + \vec{v} \times \vec{B} \]

\[ \Rightarrow \vec{E} = -\vec{v} \times \vec{B} \]

Gives \( \vec{v} = \frac{\vec{E} \times \vec{B}}{B^2} \)

Equation of \( \frac{eB}{m} \) drift at \( \vec{v}_d = \frac{\vec{E} \times \vec{B}}{B^2} \)

In general any force \( \vec{F} \) (substitute \( \vec{F} = q \vec{E} \))

\[ \vec{v}_d = \frac{\vec{F} \times \vec{B}}{qB^2} \]

For force \( \vec{F} \)
How the Radiation Belts Work

• Grad B drift = $-\mu \nabla \vec{B} \times \vec{B} / qB^2$

• Force on Parallel Motion
  – Magnetic bottle
  – Pitch angle
  – Bounce Motion

• Longitudinal Drift

• Radiation Belt Organization
  – Loss cone
Non uniform $\mathbf{B}$

Acts like a particle slowing down when it moves into $\mathbf{VB}$.

Force $=-\mathbf{u} \mathbf{V} \mathbf{B}$ force on magnetic moment $\mu$

$$\mu = \frac{E_\perp}{B} = \frac{1}{2} \frac{m v_\perp^2}{B} = \text{current} \times \text{area}$$

Assume $\nabla \propto \frac{1}{L}$ and $L \gg r_c$

Guiding center equation

$$\vec{V}_G = -U_\parallel \vec{b} + \frac{E_x \vec{b}}{B^2} - \frac{\mu \vec{V} \times \vec{b}}{g B^2} + \frac{e}{g B^2} \vec{M} \frac{v_\parallel^2 (\vec{b} \cdot \vec{V})}{B^2}$$

- $E_x \vec{b}$: Gradient drift
- $\mu \vec{V} \times \vec{b}$: Curvature drift
Parallel Motion of Guiding Center

- From Lorentz Force, looking at average motion parallel to B \( (v_{||} = \vec{v} \cdot \vec{B}/B) \)

we get this equation of motion:

\[
\frac{d}{dt} m v_{||} = e E_{||} - \mu \frac{\partial B}{\partial t}
\]

Where \( \mu = \frac{1}{2} m v_{\text{perp}}^2 / B = \text{constant when } E_{||} = 0 \)

(for derivation see http://earthweb.ess.washington.edu/bobholz/ess515/parallel_guiding_center_motion.pdf)
Magnetic Bottle

Considers a cylindrically symmetric magnetic field with field lines as

\[ B \]

\[ B_0 \]
Suppose you start a particle on the axis at the place where field strength is $B_0$. Suppose the velocity of the particle makes an angle $\alpha$ with respect to the magnetic field. What happens?

$\alpha$ = pitch angle

$v_{\parallel} = v \cos \alpha$

$v_{\perp} = v \sin \alpha$

$v_{\parallel}$ carries the particle into a region of larger field strength some time later we have $v_{\parallel}$ to right $v_{\perp}$ into paper

at the particle $\vec{B}$ has a component $B_1$ that is parallel to the guiding center field and a component $B_2$ that is perpendicular.
So we have

\[ \times \rightarrow v_{\|} \]

\[ v_{\perp}(\text{into}) \]

Consider components of \( \vec{v} \times \vec{B} \) (Lorentz Force)

\( v_{\|} B_2 \) causes force that increases \( v_{\perp} \)
\( v_{\perp} B_2 \) decreases \( v_{\|} \)

Thus, \( v_{\|} \) decreases while \( v_{\perp} \) increases.

Since \( v_{\|}^2 + v_{\perp}^2 = \text{constant} \) (energy conserved),
after a while \( v_{\|} \) will be zero. But
\( v_{\perp} B_2 \) continues to act, so \( v_{\|} \) changes sign.

i.e.: Particle enters magnetic field a certain distance, stops, turns around and comes out!
This is called **Mirroring**
We want to know the point where the particle turns around, or the mirror point.

Equation for $v_\perp$ is $\mu = \text{constant}$, 
$$ \frac{1}{B} \frac{d\mu v_\perp^2}{dt} = \text{const.} $$

If particle moves into region where $B$ is larger, then its $v_\perp$ must get larger too.

**Question:** How far into the region of increasing magnetic field will a particle penetrate if it starts with pitch angle $\phi_0$ at $B = B_0$?

$$ m = \frac{1}{B_0} \frac{\mu v_\perp^2 \sin^2 \phi_0}{\sin \phi_0} = \frac{1}{B} \frac{\mu v_\perp^2 \sin^2 \phi_0}{\sin \phi_0} \quad (v_\perp = \text{initial}) $$

As it penetrates, $B$ increases, so does $\sin \phi_0$ until $\phi_0 = 90^\circ$. The mirror point $\phi = 90^\circ$.

$$ u_\parallel = u \cos \phi = 0 $$

$$ u_\perp = u \sin \phi = u = \text{all energy, no } \frac{\mu v_\perp^2}{B} $$

Define $B_m$ at this point to be $B_m$, we have

$$ \frac{\sin^2 \phi_0}{B_0} = \frac{1}{B_m} $$

or

$$ B_m = \frac{B_0}{\sin^2 \phi_0} $$

All particles of any energy, charge, mass will mirror at same point $B_m$ if they start with same $\phi_0$.!
Another Example of Guiding Center Motion

Particles in a Magnetic Dipole Field

The Radiation Belt

As the charged particles move away from the equator, the inner region of increasing B field can mirror and bounce back and forth.

Additionally, the guiding center drifts around the earth in longitude due to gradient and curvature drifts.

I Gyrating

\[ \text{Gyration} = \text{gyropioid} = \frac{2\pi}{2} = \frac{2\pi \cdot m}{Be} \approx 10^{-2} \text{ sec} \text{ proton at low}\]

\[ 10^{-3} \text{ sec} \text{ at low}\]

\[ r = 10 \text{ RE} \]

\[ 10^{-2} \text{ sec} \text{ at high}\]

\[ r = 10 \text{ RE} \]
# Bounce

Bounce time \( T_{\text{bounce}} = \oint \frac{ds}{v_{11}} \)

\[
ds = \left( dr^2 + r^2 d\lambda^2 \right)^{1/2} = r_e \cos \lambda (1 + 3 \sin^2 \lambda)^{1/2} \ d\lambda
\]

\[
v_{11} = \left( v^2 - v_+^2 \right)^{1/2}
\]

but \[
\frac{v_+ (r, \lambda)}{B (r, \lambda)} = \frac{v^2 \sin \alpha_e}{B_e} = \text{const}
\]

where \( \alpha_e \) and \( B_e \) are equatorial pitch angle and field strength.

so \[
v_{11} = v \left( 1 - \frac{\sin^2 \alpha_e B (r, \lambda)}{B_e} \right)^{1/2}
\]

\[
B (r, \lambda) = \frac{M}{r^2} \left( 1 + 3 \sin^2 \lambda \right)^{1/2} \quad \text{(Dipole Field)}
\]

\[
= \frac{B_e}{\cos \lambda} \left( 1 + 3 \sin^2 \lambda \right)^{1/2} \quad \text{(because } \frac{r = r_e \cos \lambda}{\text{field line}})\text{)}
\]

numerically integrating \[
T_{\text{bounce}} \sim 4 \frac{r_e}{v} T (\alpha_e)
\]

where \( T (\alpha_e) = 1.3 - 1.5 \times \sin \alpha_e \) (noro strong dependence on \( \alpha_e \))

for \( v = \infty \) and \( T (\alpha) = 1 \)

\[
\begin{array}{cccc}
r_e & 1.5 \ r_e & 3 \ r_e & 6 \ r_e \\
T_{\text{bounce}} & 135 & 1265 & 1525
\end{array}
\]
III Longitudinal Drift

for $J = 0 = \pi \times 3$

$$V_G = \frac{\beta \times 0.3}{eB^3} (E_{\perp} + 2E_{\parallel})$$

integrate around drift path of $360^\circ$ longitude

gives $T_{\text{drift}} \approx \frac{44}{Re \times E(\text{mev})}$ minute

That is, $re \approx LRe \approx L$ earth radii

For $E = 0.1 \text{ Mev}$ and $Re = 10 Re$

$T_{\text{drift}} \approx 44$ minutes

So, in general

$T_{\text{drift}} > T_{\text{bounce}} > T_{\text{gyro}}$

Neglecting scattering and plasman instabilities

Particles can be trapped forever

In practice some species at some energies

are trapped for 100 yrs.
Fig. 6. Contours of constant adiabatic gyration, bounce, and drift frequency for equatorially mirroring particles in a dipole field. Adiabatic approximation

L is the place the dipole field line crosses the equator, in units of Earth Radii
Radiation Belt organization

For a dipole $B$-field with $M = \text{dipole moment of Earth}$

\[ \vec{M} = \iiint \vec{m}(\vec{r}, \vec{v}, t) \, d^3v = \text{magnetic moment/volume} \]

\[ \equiv \text{magnetization} \]

Magnetic field $\vec{B} = -\nabla \psi$

where $\psi = -\frac{\vec{M} \cdot \nabla (\frac{1}{r})}{r}$ for dipole

in spherical coordinates:

$B_r = \frac{B}{r} \psi = -2M \frac{\sin \phi}{r^3}$

$B_\phi = \frac{1}{r \sin \phi} \frac{d}{d \phi} \psi = 0$

$B_\lambda = -\frac{1}{r \sin \phi} \frac{d}{d \lambda} \psi = M \frac{\cos \phi}{r^3}$

$|\vec{B}| = \sqrt{B_r^2 + B_\phi^2 + B_\lambda^2} = \frac{M}{r^3} \left( 1 + 3 \sin^2 \lambda \right)^{1/2}$

From Pages 3.30/3.31 can write

\[ \frac{dr}{r} = 2 \frac{d (\cos \lambda)}{\cos \lambda} \]

Integrate to get

$\phi = \phi_0 \quad r = r_0 \cos^2 \lambda$

Now let $L = \frac{r_0}{Re} \quad \text{marks Equatorial distance}$

so $r = L Re \cos^2 \lambda$
or, to put it another way, the latitude at the surface is given by \( r = R_e \) so \( \cos^2 x = \frac{1}{2} \)

This is definition of Invariant Latitude

\( L \) can be defined more carefully for a distorted field.

So, for a given \( L \) shell, most particles will be undergoing 3 distinct motions—gyration, bounce + DB drift—but its mean point is \( \leq 1 \text{ Re} \). They will be lost due to scattering from atmosphere.

Actually, the loss altitude \( \approx \text{Re} + 100 \text{km} \) where probability of scattering becomes high.
Which particles are lost?

\[ d\Omega \text{ solid angle} \]
\[ \text{loss cone} \]

if minm point \( \theta \leq R + 100 \text{ km} \), then find this \( \delta \theta \)

using \( \mu = \text{constant} = \frac{u^2 \sin \delta \theta}{\beta e} \)

find \( \delta \theta \) such that \( \delta \theta = 90^\circ \) at 13 for \( r = R + 100 \text{ km} \)
we find that \( \delta \theta \sim 3^\circ \) at \( L = 6 \)
so MOST particles are trapped.

show pitch angle distributions along field line.
L-shell drift: How can you tell
the drift motion always returns guiding
center to starting point?

Answer: if energy is
conserved then
with μ = constant, if
start from r_i where B = B_i

\[ μ = \frac{E}{B_i} \] at r_i

if returned at r ≠ r_i, then
so μ ≠ constant

L defines a closed shell for perfect dipole

equatorially mirroring

Then \( E = E_1 \) always

assume

\[ \text{equatorially mirroring} \]

Then \( E = E_1 \) always
Advanced subject

For details see adiabatic_invariants.pdf

Adiabatic Invariant

\[ I(\vec{3}, \vec{8}) = \oint \vec{p} \cdot d\vec{q} \cdot \frac{2g(\vec{3}, \vec{8})}{2\pi} \]

usually abbreviated

\[ I = \oint \vec{p} \cdot d\vec{q} = \text{then} \oint d^3p \cdot d^3q \quad \vec{z} = \text{constant} \]

but \( \oint d^3p \cdot d^3q \) has same value for any canonical \( \vec{p}, \vec{q} \)

\[ \therefore \text{since transformation from unperturbed to perturbed } \vec{p}, \vec{q} \text{ is canonical} \]

\[ \oint d^3p \cdot d^3q = \oint d^3p \cdot d^3q = \oint \vec{p} \cdot d\vec{q} \]

actual \quad \text{unperturbed final} \quad \text{unperturbed initial} \quad \vec{z} = \text{constant}