2. Basic Equations

2.1. Fluids

The scales of phenomena that we will be interested in are quite large compared to those that modern physicists typically concentrate on. This means that quantum effects and complications such as relativity or Heisenberg's uncertainty principle are not important. Although some of the phenomena that we will discuss have been reasonably well understood only in the last three decades, the *physics* underlying them was well known to scientists of the 19th century. In fact, many of the key ideas go back to classical time. This should not be surprising, because early scientists spent most of their time trying to understand the world they could easily observe around them.

GFD is a branch of continuum mechanics. This is a statement that the scale is large compared to intermolecular distances. A continuum is a material whose properties such as density, temperature etc. vary smoothly except at a finite number of discrete boundaries. Density (ρ) is the mass per unit volume. In real materials, ρ ceases to be a smooth function if the test volume becomes too small (see Figure 1). We therefore always consider spatial averages over volumes that are sufficiently large that the average material properties are indeed continuous. This is not a particularly onerous restriction, because the minimum volume is very small compared to the scale of most geophysical phenomena. The connection between the molecular and continuum properties of a material is the science of statistical mechanics, another branch of modern physics that we will not need to consider in detail.

A fluid is a material which cannot withstand any tendency by applied forces to deform it in a way which leaves its volume unchanged. Consider forces acting normal to the surface of a small volume of fluid that is often called a fluid element or a fluid parcel. The force per unit area is called the stress. The stresses can always be broken down into an isotropic part which is independent of direction and a deviatoric part (see Figure 2(a)). The isotropic part squeezes the fluid (i.e. trys to change its volume). A fluid can build up internal stress that oppose this squeezing and so an isotropic stress (called the hydrostatic pressure) can exist in a fluid at rest. A fluid will deform forever as long as any deviatoric stress exists. Stress that acts tangential to the surface of a fluid element is called a shear stress. Shear stresses are always deviatoric and can be expressed in terms of non-isotropic normal stresses by a suitable change of coordinate axes (see Figure 2(b)). In fluids, it is usually easiest to break the stresses acting into the pressure (the isotropic normal stress) and the shear (the tangential stress).

There are basically two kinds of fluid:

(a) Gas - A loose configuration of molecules which interact only by (completely elastic) collision. Stresses in a gas are the result of the transfer of momentum from molecule to molecule when they collide. Gasses can be held together only by walls or gravity.

(b) Liquid - A much closer configuration of molecules which act on each other without necessarily colliding. Liquids commonly consist of molecular chains and stresses in a liquid involve not only momentum transfer in collisions, but also the breaking and welding of intermolecular bonds.

Gasses are more compressible than liquids. We shall see that compressibility effects are important only when the fluid velocity is comparable to the sound velocity. Thus the compressibilities of gasses are still too small to be important in most geophysical contexts. The only practical difference between liquids and gasses is the ability of a liquid to form a free surface. Because of surface tension, a free surface acts like a rubber sheet laid over the surface of the liquid. This sheet can confine the liquid in the absence of walls. Because the effects of compressibility can be ignored, the fluid mechanics of the ocean and atmosphere have many similarities. The ocean, of course is primarily a liquid, while the atmosphere is primarily a gas. The fact that the ocean sometimes includes solid water (ice) has dynamical consequences, but they are indirect and generally local. On the other hand, the presence of liquid droplets and ice particles in the atmosphere and the latent heat associated with phase transitions between solid, liquid and gaseous water have major global impacts on the dynamics of the atmosphere and make atmosphere dynamics somewhat more complicated than ocean dynamics..

2.2. Equations of motion

Fluids are governed by the ordinary laws of mechanics: (a) The conservation of mass and (b) The conservation of momentum. The conservation of mechanical energy can be derived from the conservation of momentum, so we need to consider energy conservation separately only when other forms of energy (thermal or electromagnetic) are important. To the above two laws, we must also add (c) The equation of state, which relates the density of the fluid to thermodynamic variables such as pressure and temperature. In most of this course we will restrict our attention to incompressible fluids whose density is independent of all thermodynamic variables (but not necessarily independent of position).

2.2.1. Lagrangian and Eulerian descriptions

Before writing down the equations corresponding to the physical laws, we need to discuss how we are going to describe the motion of a fluid. In ordinary mechanics, we are used to dealing with the action of forces on individual particles and it is natural to describe their trajectory as a function of time. We can do the same thing for a fluid if we describe the position **r** of each fluid element as function of time. The situation is complicated, however, by the very large number of fluid elements involved. Thus we need a way to label each element.

One way is to give the function

$$\mathbf{r} = \mathbf{r}(\mathbf{x}, t)$$

where **x** is the position that the fluid element had at time $t = 0$. This is called the Lagrangian description of the motion. Note that **x** in the above function is independent of time. Keeping **x** fixed is equivalent to following the behavior of a specific fluid element.

You can imagine the Lagrangian description as painting coordinate lines on the fluid at time zero and then watching how these lines deform as the fluid moves. You can also think of the Lagrangian point of view as being what an observer sees when he or she moves with a fluid element. An example of a Lagrangian measurement in the ocean would be the position of a float cast adrift in a current. The temperature of the water adjacent to the drifting float would also be a Lagrangian measurement. If you take the time derivative of $\mathbf{r}$ (holding $\mathbf{x}$ fixed), you get

$$\frac{D\mathbf{r}}{Dt} = \mathbf{u}_L(\mathbf{x}, t)$$

This is called the Lagrangian velocity because it is the velocity of the fluid element that was at $\mathbf{x}$ at time 0. I have used the symbol $\frac{D}{Dt}$ to emphasize the fact that the derivative is calculated from the point of view of an observer following a specific fluid element.

Instead of considering the velocity to be a function of which fluid element we look at, it is often more convenient to consider it as a function of the present position. In fact, we can completely describe the motion of the fluid by giving the velocity everywhere at a given instant in time. This is the Eulerian description

$$\frac{D\mathbf{r}}{Dt} = \mathbf{u}_E(\mathbf{r}, t)$$

You can think of the Eulerian description as an instantaneous snapshot of the velocity field. You can also envision the Eulerian point of view as making measurements as a function of time at fixed points in space. Thus an Eulerian current measurement would involve observing the rate of flow past a moored buoy or stationary platform.

The Eulerian velocity $\mathbf{u}_E$ and the Lagrangian velocity $\mathbf{u}_L$ are numerically equal at every point. However they depend on different variables. The Lagrangian description requires knowing the path of each fluid element; the Eulerian description requires knowing the velocity of each fluid element at every point in space. Obviously the two descriptions are related. One gets the Lagrangian description from the Eulerian description by integrating the velocity field with respect to time. The initial position of each fluid element element enters as the constant of integration. One gets the Eulerian description from the Lagrangian description by differentiating with respect to time and inverting the path $\mathbf{r}(\mathbf{x}, t)$ to eliminate $\mathbf{x}$ in the velocity. You will have the opportunity to do this in a problem.

2.2.2. The substantive derivative

A tricky point that is primarily responsible for making fluid mechanics more complicated than ordinary mechanics is that the conservation laws refer to the fluid inside a small test volume. The boundary of this volume is attached to the molecules of the fluid and no mass is ever allowed to cross its surface. Thus the physical laws of mechanics are from the Lagrangian point of view. We shall often need to know the form the laws take from an Eulerian point of view. This requires relating the time derivative of a function measured by an observer moving with the fluid to that measured by an observer fixed in space. The total differential of a function $F(x, y, z, t)$ is

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy + \frac{\partial F}{\partial z}\,dz + \frac{\partial F}{\partial t}\,dt$$

Now suppose that an observer moves at a velocity $\mathbf{u} = u\hat{\mathbf{x}} + v\hat{\mathbf{y}} + w\hat{\mathbf{z}}$ relative to another observer (who we shall define as fixed for this discussion). (Note the naming convention that I have used for the components of **u**; u is the x component and the other components are in alphabetical order. I will use this convention throughout these notes.) Both observers make measurements of the changes in F. By the definition of the partial derivative, the fixed observer measures $\frac{\partial F}{\partial t}$. The moving observer, however, will change his x position by the amount $dx = udt$ in the time dt and will see a change $\frac{\partial F}{\partial x}\,dx = \frac{\partial F}{\partial x}\,udt$ in F due to this x displacement. Considering all three components of motion, plus the change in F seen by the fixed observer, the total change in F seen by the moving observer is

$$dF = \left[\frac{\partial F}{\partial x}\,u + \frac{\partial F}{\partial y}\,v + \frac{\partial F}{\partial z}\,w + \frac{\partial F}{\partial t}\right] dt$$

Using the definition of the vector dot product and

$$\nabla = \hat{\mathbf{x}}\,\frac{\partial}{\partial x} + \hat{\mathbf{y}}\,\frac{\partial}{\partial y} + \hat{\mathbf{z}}\,\frac{\partial}{\partial z}$$

the rate of change of F in the time dt seen by the moving observer is

$$\frac{DF}{Dt} = (\mathbf{u}\cdot\nabla)F + \frac{\partial F}{\partial t}$$

One again, I have used the symbol $\frac{D}{Dt}$ to emphasize the fact that this is a derivative measured by a moving observer. Usually, **u** will be the velocity of a fluid. Thus $\frac{DF}{Dt}$ is the time derivative measured by an observer moving with the fluid. For this reason it is often called the substantive or substantial or advective or moving or Lagrangian derivative. If we rewrite the above relation

$$\frac{\partial F}{\partial t} = \frac{DF}{Dt} - (\mathbf{u}\cdot\nabla)\,F$$

the left side is now the time derivative seen by the fixed observer. The first term on the right is the intrinsic change of F in the moving system. The second term is the additional change in F due to the fact that spatial variations of F are being swept by the fixed observer (see Figure 3).

2.2.3. Conservation of mass

Suppose that V is a volume whose boundary moves with the fluid of density ρ. If V is sufficiently small, we can assume that ρ is constant inside V without loss of generality. The total mass inside the volume is $M = \rho V$. If M is conserved as the fluid moves,

$$0 = \frac{DM}{Dt} = \frac{D}{Dt}(\rho V) = \frac{D\rho}{Dt} V + \rho \frac{DV}{DT}$$

Dividing by V (which is never 0) we obtain

$$\frac{D\rho}{Dt} + \frac{\rho}{V} \frac{DV}{Dt} = 0$$

Now further suppose that V is a small cube, whose sides are of length X, Y and Z. If the velocity varies in the x direction, it will have two effects (see Figure 4). If u is larger by an amount δu at the right end of the cube than at the left, it will stretch the length of the cube. After a time δt, the length of the cube will be

$$X + \delta X = X + (u + \delta u)\delta t - u\delta t = X + \delta u \delta t$$

If X is short, we are justified in approximating the velocity variation by the first two terms in its Taylor series. Then $\delta u = \frac{\partial u}{\partial x} X$ and

$$X + \delta X = (1 + \frac{\partial u}{\partial X} \delta t) X$$

On the other hand, if v or w vary in the x direction they will result in a rotation of the cube, but to first order, there will be no change in the length of the side.

If we apply the same reasoning to each side of the cube, the volume of the cube after a time δt is

$$V + \delta V = (X + \delta X)(Y + \delta Y)(Z + \delta Z) = (1 + \frac{\partial u}{\partial x} \delta t)(1 + \frac{\partial v}{\partial y} \delta t)(1 + \frac{\partial w}{\partial z} \delta t)\ XYZ$$

Performing the multiplication on the right and dropping terms multiplied by $(\delta t)^2$ (which are much smaller than those multiplied by δt, when δt is small) gives

$$V + \delta V = [1 + (\nabla \cdot \mathbf{u})\delta t]\ V$$

where the divergence of **u** is defined by

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Finally, for small δt

$$\frac{1}{V}\frac{DV}{Dt} = \nabla \cdot \mathbf{u}$$

and the conservation of mass equation becomes

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{u} = 0$$

Using the definition of the substantive derivative of ρ, this can be re-written.

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla\rho + \rho\nabla \cdot \mathbf{u} = \frac{\partial \rho}{\partial t} + \nabla \cdot \rho\mathbf{u} = 0$$

which is the Eulerian form of the mass conservation equation.

For an incompressible fluid, the density is constant as the fluid moves. This implies that

$$\frac{D\rho}{Dt} = 0$$

or

$$\nabla \cdot \mathbf{u} = 0$$

The first relation is the Lagrangian form of the conservation of mass for an incompressible fluid, while the second is the Eulerian form.

2.2.4. Conservation of momentum

The momentum of a particle with mass M is defined to be $M\mathbf{u}$. Newton's Second Law simply states that rate of change of momentum is equal to the forces acting on the particle. When the particle is a volume which always encloses the same mass of fluid we can write this law

$$\frac{D}{Dt}(M\mathbf{u}) = M\frac{D\mathbf{u}}{Dt} = \mathbf{F}$$

where $\mathbf{F}$ is the vector sum of all forces acting. Dividing by the volume, this can be re-written

$$\rho\frac{D\mathbf{u}}{Dt} = \rho(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla\mathbf{u}) = \mathbf{f}$$

where **f** is now the vector sum of all forces per unit volume.

Note that in the Eulerian form, the scalar expression $\mathbf{u} \cdot \nabla$ operates on the vector **u**. If **u** is expressed in a particular coordinate system, $\mathbf{u} \cdot \nabla$ operates on both the components and the unit vectors. In Cartesian coordinates in which the unit vectors are constant as a function of position, the result is simply evaluated by applying the operator to each component of **u**. In curvilinear coordinates in which the unit vectors vary with position (such as the azimuthal unit vector $\hat{\phi}$ of cylindrical coordinates), one must be careful, because $\mathbf{u} \cdot \nabla \hat{\phi}$ is not zero. (See Tritton (2nd Edition) pages 60-61 for expanded versions of $\mathbf{u} \cdot \nabla \mathbf{u}$ in several standard coordinate systems.) The added terms arising from the operation of the advective part of the derivative on the curvilinear unit vectors correspond physically to the accelerations necessary to keep the fluid element following the curved coordinate line. Note also that $\mathbf{u} \cdot \nabla \mathbf{u}$ involves the product of the velocity with itself. Therefore this term is non-linear. It is responsible for much of the difficulty of fluid mechanics.

2.2.5. Body forces

The force per unit volume on the right side of the momentum conservation equation is the vector sum of all forces acting on the volume. These are of two basic kinds. Those that act directly throughout the interior of the volume and those that act on the surface and thus on the interior only through molecular interactions. The basic body forces are those of gravity and electromagnetism.

The gravity term can be written

$$\mathbf{f}_{gravity} = \rho \mathbf{g} = \rho g \hat{\mathbf{z}} = \rho \nabla \phi_{gravity}$$

where **g** is called the acceleration of gravity, which we will almost always define to be in the $\hat{\mathbf{z}}$ direction. The sign of the potential $\phi_{gravity}$ is often defined to be the negative of the one I have used.

Electric forces play a negligible role in the ocean. However they can be important in other geophysical contexts and measurement of the electric field produced when conducting sea water moves through the Earth's magnetic field is a new frontier in ocean current measurement with a great deal of promise. The total electric field measured relative to a moving conductor is

$$\mathbf{E} = \mathbf{E}' + \mathbf{u} \times \mathbf{B}$$

where $\mathbf{E}'$ is the electric field in the absence of the moving conductor and **B** is the magnetic field. It is the second term that can give direct information about ocean currents. Three applications have been tested. The oldest involved towing an antenna several tens of meters long behind a ship. Although electric fields were measured, the experiment failed because the background $\mathbf{E}'$ due to ionospheric and magnetospheric currents was not removed with sufficient accuracy. However, a cable stretching from Florida to the Bahamas under the Gulf Stream has been used to make a long time series of the transport of this very important ocean current. In this case, the non-oceanic signal is removed by cross-correlation with a distant magnetic observatory. Recent interest in making long term transport measurements on a global scale for studying global climatic change combined with the shift of trans-oceanic communication to satellites and fiber optics and the consequent availability of many cables has greatly increased interest in this technique. Another recent application involves deployment of instruments with antennas only several meters long up to a year on the ocean bottom to make long period measurements of oceanic motions. A distinct advantage of this electromagnetic technique is that it is an Eulerian measurement that averages the current over a fairly large volume around the instrument.

If the fluid contains a positive charge density, ρ_e, then the fluid experiences a force

$$\mathbf{f}_e = \rho_e \mathbf{E}' + \mathbf{j} \times \mathbf{B}$$

where $\mathbf{j} = \rho_e \mathbf{u}$ is the electric current density. The second term in this relation is called the Lorentz force. When the fluid conductivity is very large as it is in the Earth's core and in space plasmas, $\mathbf{B}$ depends on $\mathbf{u}$ and $\mu_0 \mathbf{j} = \nabla \times \mathbf{B}$, where μ_0 is the permeability of free space. Then the Lorentz force becomes another non-linear term in the conservation of momentum. It is this non-linearly that makes magneto-hydrodynamics (MHD) more difficult than GFD. I will not consider these forces further because they are too small to have dynamical significance in the ocean. I will also ignore the force on a fluid containing a suspension of magnetic dipoles subject to a magnetic field because I know of only one remotely geophysical application: the motion of an ooze containing a high concentration of magnetotatic bacteria.

2.2.6. Surface forces

Consider first the normal force acting over the surface area A of a small fluid volume V (see Figure 5). Let $\hat{\mathbf{n}}$ be a unit vector normal to the surface and pointing outwards. By convention, pressure is defined to be positive when it points inwards. Thus the $\hat{\mathbf{x}}$ component of the normal force per unit area at each point on the surface is $\hat{\mathbf{x}} \cdot - p\hat{\mathbf{n}}$. Integrating over the entire surface A, the total force on V in the $\hat{\mathbf{x}}$ direction is

$$F_x = \int \hat{\mathbf{x}} \cdot - p\hat{\mathbf{n}} \; dA$$

Now Gauss' Theorem states that for any vector field $\mathbf{Q}$ and any closed surface A with outward unit normal $\hat{\mathbf{n}}$,

$$\int \hat{\mathbf{n}} \cdot \mathbf{Q} \; dA = \int \nabla \cdot \mathbf{Q} \; dV$$

where V is the volume enclosed by A. Letting $\mathbf{Q} = - p\hat{\mathbf{x}}$, we then immediately have

$$F_x = - \int \nabla \cdot \; p\hat{\mathbf{x}} \; dV = - \int \frac{\partial p}{\partial x} \; dV$$

The force per unit volume as the volume shrinks in size is just the integrand $-\dfrac{\partial p}{\partial x}$. If we apply the same argument to the other two coordinate directions we finally conclude that the total pressure force per unit volume must be

$$\mathbf{f} = -\frac{\partial p}{\partial x}\hat{\mathbf{x}} - \frac{\partial p}{\partial y}\hat{\mathbf{y}} - \frac{\partial p}{\partial z}\hat{\mathbf{z}} = -\nabla p$$

The physics of this result is not at all surprising. It simply states that there is a force on the fluid from regions of high pressure to those of low.

Turning now to the tangential force. Consider the following simple experiment: A layer of fluid is confined between two horizontal parallel plates (see Figure 6). A force per unit area τ_{xz} (called a shear stress) in the $\hat{\mathbf{x}}$ direction is applied tangential to the top plate. The bottom plate is held fixed. For a wide variety of fluids, it is empirically observed that the velocity in the fluid will decrease linearly from the its maximum at the top plate to zero at the bottom plate. This is called Couette flow. The relation

$$\tau_{xz} = \mu \frac{\partial u}{\partial z}$$

defines μ, the dynamic viscosity of the fluid. $\dfrac{\partial u}{\partial z}$ is called the strain rate and those familiar with elasticity will note the strong similarity between this relation and Hooke's Law for shearing of an elastic solid. For so-called Newtonian fluids such as liquid water and air, μ is independent of the the shear stress or the strain rate. However many fluids of geophysical interest such as ice and rocks below their melting temperature are non-Newtonian and have viscosities that depend on stress and strain rate. Most of these materials become softer as the strain rate increases. Note that the linear velocity gradient of Couette flow does not depend on the fluid being Newtonian. It is simply sufficient that there be a unique relation between stress and strain rate because the stress, strain rate and hence viscosity are all constant within the flow. It should be pointed out, however, that Couette flow is not a unique solution to the problem when the fluid is non-Newtonian and the transients associated with getting the top plate moving may result in a different flow pattern. Ocean water is Newtonian to very high accuracy and we will ignore non-Newtonian effects in this course.

In order to incorporate the shear (which henceforth we will call viscous) force into the conservation of momentum, we need to to find the equivalent force per unit volume. To do this, consider a horizontal tabular region within a shearing fluid (see Figure 7). If the tablet is thin compared to its horizontal dimensions, the contributions from the vertical edges are negligible and the force on the tablet in the $\hat{\mathbf{x}}$ direction is

$$F_x = \int_{top} (\tau_{xz})_{top}\, dA + \int_{bottom} (\tau_{xz})_{bottom}\, dA$$

Since the shear force acting on the fluid inside the tablet and the outward unit normal are both of opposite signs on the top and bottom faces, this can be written

$$F_x = \int \tau_{xz}\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} \; dA$$

where now the integral is over both horizontal surfaces of the tablet and $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ or $-\hat{\mathbf{z}}$ is the appropriate outward unit normal. If we now let $\mathbf{Q} = \tau_{xz}\hat{\mathbf{n}}$ and apply Gauss' Theorem, we obtain

$$F_x = \; = \int \nabla \cdot \tau_{xz}\hat{\mathbf{n}} \; dV = \int \frac{\partial \tau_{xz}}{\partial z} \, dV$$

In the limit as the tablet becomes very small, the force per unit volume for a Newtonian fluid becomes

$$f_x = \frac{\partial \tau_{xz}}{\partial z} = \mu \frac{\partial^2 u}{\partial z^2}$$

Repeating the argument for all possible orientation of forces and tablets within a shear flow, the total viscous term becomes

$$\mathbf{f}_{viscous} = \mu \nabla^2 \mathbf{u}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is the Laplacian operator. Note that, just as for the advective term in the fluid acceleration, when this scalar operator is applied to a vector expressed in curvilinear coordinates, it operates on the unit vectors as well as the components. (See Tritton, pages 60-61 for the expanded viscous terms in cylindrical an spherical coordinates.)

This derivation of the viscous term is neither complete nor rigorous and in fact does not even give quite the correct answer. A more rigorous derivation discussed in the Appendix to Chapter 5 in Tritton (2nd Ed.) concludes that the $\hat{\mathbf{x}}$ component of the viscous force is

$$\frac{\partial}{\partial x}\left[2\mu \frac{\partial u}{\partial x} + \lambda \nabla \cdot \mathbf{u}\right] + \frac{\partial}{\partial y}\left[\mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})\right] + \frac{\partial}{\partial z}\left[\mu(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z})\right]$$

Note that this involves a second viscosity coefficient λ associated with compression. Fortunately, as already noted, we are justified in assuming that the fluids we are considering are incompressible and that $\nabla \cdot \mathbf{u} = 0$. It is then easily shown that this result reduces to the simpler one above. Even in supersonic flow, it is only the gradient of the compression that comes into the viscous dissipation. This is significant only within a shock wave.

2.2.7. Effects of rotation

The final topic that we need to consider is the fact that we commonly make measurements of oceanic or atmospheric flow from the surface of the Earth. Because the Earth rotates, a point on the surface is being continuously accelerated as it follows a curved path around the pole of rotation. Let **u** be the velocity of the fluid observed from the surface of the Earth and let Ω be a vector which points along the rotation axis towards the star Polaris and has a length equal to the angular rotation rate of the Earth. The velocity of the observer due to the Earth's rotation is $\mathbf{u}_r = \Omega \times \mathbf{r}$, where **r** is a vector form the center of the Earth to the observer (see Figure 8(a)). Thus the velocity of the fluid measured by an observer who is outside the Earth and does not rotate with it is

$$\mathbf{u}' = \mathbf{u} + \Omega \times \mathbf{r}$$

which we can write

$$\frac{D'\mathbf{r}}{Dt} = \frac{D\mathbf{r}}{Dt} + \Omega \times \mathbf{r}$$

Noting that the vector **r** is the same for both observers, we see that

$$\frac{D'}{Dt} = \frac{D}{Dt} + \Omega \times$$

allows us to calculate time derivatives in the non-rotating frame given time derivatives in the rotating one. We now apply this result to the fluid acceleration observed in the inertial (non-rotating) frame:

$$\frac{D'\mathbf{u}}{Dt} = \frac{D'^2\mathbf{r}}{Dt^2} = \left(\frac{D}{Dt} + \Omega \times\right)\left(\frac{D}{Dt} + \Omega \times\right)\mathbf{r}$$

$$= \frac{D^2\mathbf{r}}{Dt^2} + \Omega \times \frac{D\mathbf{r}}{Dt} + \frac{D\Omega}{Dt} \times \mathbf{r} + \Omega \times \frac{D\mathbf{r}}{Dt} + \Omega \times \Omega \times \mathbf{r}$$

Assuming that $\frac{D\Omega}{Dt} = 0$ (a very good approximation for the Earth), this reduces to

$$\frac{D'\mathbf{u}}{Dt} = \frac{D\mathbf{u}}{Dt} + 2\Omega \times \mathbf{u} + \Omega \times \Omega \times \mathbf{r}$$

The term $2\Omega \times \mathbf{u}$ is is often shifted to the force side of the equation. The component of this Coriolis ''force'' tangential to the surface of the Earth is perpendicular to and to the right of the velocity in the Northern Hemisphere, to the left in the Southern Hemisphere. It is responsible for many of the interesting phenomena of GFD. It is not widely appreciated that there is also a component of the Coriolis force perpendicular to the surface of the Earth, whose magnitude is $2\Omega v \sin\theta$, where v is the magnitude of the velocity in the eastward direction and θ is the colatitude. This vertical force is called the Eotvos effect and is so small compared to the acceleration of gravity that it is generally ignored. However, it can be significant compared to the anomalous gravity field associated with local mass variations, so that one needs to account for it when making a gravity survey from a moving vehicle such as a ship or airplane.

When the term $\Omega \times \Omega \times \mathbf{r}$ is shifted to the force side of the equation, it is called the centrifugal ‘‘force’’. It depends only on the geometry of the observer’s position relative to the rotation axis. In cylindrical coordinates R, ϕ, z (see Figure 8(b)), the centrifugal force becomes

$$\mathbf{f}_{centrifugal} = \rho\Omega^2 R\hat{\mathbf{R}} = \rho\nabla(\frac{1}{2}\Omega^2 R^2) = \rho\nabla\phi_{centrifugal}$$

Thus the centrifugal force and the gravitational force are both gradients of scalars. By defining a new potential

$$\phi_{total} = \phi_{gravity} + \phi_{centrifugal}$$

and a new effective gravity

$$\mathbf{g}_{effective} = \nabla\phi_{total}$$

we need no longer specifically consider the centrifugal force. You should note that a gravity meter on the surface of the Earth actually measures $\mathbf{g}_{effective}$, so that standard gravity formulas already include the effect of the centrifugal force.

2.2.8. The Navier-Stokes Equation

Pulling all parts of the conservation of momentum equation together, and dividing by ρ we finally obtain

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + 2\Omega\times\mathbf{u} = -\frac{1}{\rho}\nabla p - g\hat{\mathbf{z}} + \nu\nabla^2\mathbf{u}$$

where $\nu = \dfrac{\mu}{\rho}$ is called the kinematic viscosity and $\mathbf{g}$ is the total effective gravity.