3. Simple Solutions

In this chapter we consider several applications of the Navier-Stokes equation in which the difficult terms are identically zero.

3.1. Static Equilibrium

The very simplest situation is when $\mathbf{u} = 0$. The Navier-Stokes equation then reduces to

$$\nabla p = - \rho g \hat{\mathbf{z}}$$

Whose components are

$$\frac{\partial p}{\partial x} = 0 \qquad \frac{\partial p}{\partial y} = 0 \qquad \frac{\partial p}{\partial z} = - \rho g$$

The first two equations imply that $p = p(z)$. Assuming that g is constant (a very good approximation), the third then implies that only $\rho = \rho(z)$ can be compatible with no motion. Alternatively, we can rewrite the hydrostatic equation using the gravitational potential.

$$\nabla p = \rho \nabla \phi$$

The curl of a gradient is always zero. Thus the left side of this equation is simply

$$\nabla \times \nabla p = 0$$

The chain rule for the curl of a scalar times a vector applied to the right side gives

$$\nabla \times \rho \nabla \phi = \nabla \rho \times \nabla \phi + \rho \nabla \times \nabla \phi$$

Thus

$$\nabla \rho \times \nabla \phi = 0$$

This equation implies that the surfaces of constant ρ must be parallel to the equipotentials. This is clearly the case when $\rho = \rho(z)$ and $\mathbf{g} = g\hat{\mathbf{z}} = \nabla(gz)$ because the surfaces of constant density and the equipotentials are both horizontal and we have the same result deduced above.

As a slightly more complicated illustration, consider the shape of the free surface of water in a vertical cylindrical container rotating about its axis (see Figure 9). After we have waited long enough that any transient motions associated with spinning up the fluid have died away, $\mathbf{u}$ in a reference frame rotating with the container is zero. If the water is of constant density, the only place where the the density gradient is non-zero is at the free surface. There it is perpendicular to the surface and points down. Thus the shape of the free surface must be an equipotential. In the rotating system, the total effective gravity potential is $\phi = - gz + \frac{1}{2}\Omega^2 R^2$, where R is the horizontal distance from the rotation axis.

Choosing the equipotential $\phi = 0$, we must have $gz = \dfrac{1}{2}\Omega^2 R^2$ and hence the shape of the free surface must be parallel to the surface

$$z = \frac{\Omega^2}{2g} R^2$$

which is a paraboloid concave upwards. This result has recently been used to make very accurate large parabolic astronomical mirrors by freezing a rotating glass melt.

Returning to the $\hat{\mathbf{z}}$ component of the static Navier-Stokes equation, we have the ordinary differential equation

$$\frac{dp}{dz} = -\rho(z)g$$

which can be directly integrated to give

$$p_{hydrostatic}(z) = p(0) - \int_0^z \rho(z)g \; dz$$

In the ocean, the constant of integration, p(0) is usually taken to be the mean atmospheric pressure. Because z is negative below the surface and $\rho(z)$ is very nearly constant, the hydrostatic pressure increases downwards at the rate of about one atmosphere every 10 meters.

Because hydrostatic pressure exists in the absence of motion and because we will generally be interested in moving fluids, its is useful to redefine the pressure in the the Navier-Stokes equation to be

$$p = p_{hydrostatic} + p'$$

It is the pressure perturbation p' that is dynamically important. For the case when $\rho = \rho_0 = constant$, substituting this into the Navier-Stokes equation and subtracting the equation

$$0 = -\frac{1}{\rho_0} \nabla p_{hydrostatic} + g\hat{\mathbf{z}}$$

yields

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho_0} \nabla p' + \nu\nabla^2\mathbf{u}$$

Thus, for a constant density fluid, the only effect of gravity is to produce the hydrostatic pressure. Otherwise, gravity becomes dynamically unimportant. In the rest of these notes, I will usually drop the prime on the pressure perturbation because it will be obvious from

context whether I am discussing the total pressure or the perturbation.

3.2. Compressibility and density

A real fluid which has constant density at atmospheric pressure can never have density quite constant as a function of depth in the presence of a gravity field. Earlier I dismissed the effects of compressibility without proof. In this section I will be more more precise about this issue and briefly discuss the density of ocean water.

Compressibility is defined by

$$\beta = \frac{1}{\rho}\frac{d\rho}{dp}$$

Thus

$$dp = \frac{1}{\beta\rho}\,d\rho$$

$1/\sqrt{\beta\rho}$ has the units of velocity and can be shown to be the speed of sound c. If we divide both sides of this equation by dz and use the Navier-Stokes equation with $\mathbf{u} = 0$, we obtain

$$\frac{dp}{dz} = c^2\frac{d\rho}{dz} = -\rho g$$

When c is constant, dividing by ρ separates the variables and makes this equation an exact differential

$$\frac{d\rho}{\rho} = -\frac{g}{c^2}\,dz$$

which can be immediately integrated to give

$$\ln\rho = \ln\rho_0 - \frac{g}{c^2}\,z$$

The integration constant, $\ln\rho_0$ is usually defined as the log of the density at the air-water interface. Exponentiating finally gives the relation for the density variation in a static, constant compressibility fluid under the influence of a gravity field:

$$\rho = \rho_0 e^{-\frac{z}{H}}$$

where $H = \dfrac{c^2}{g}$ is called the scale height. For the ocean, $c \approx 1500$ m/s and $H \approx 250$ km. This is obviously very large compared to the depth of the ocean, which is typically 4 km and nowhere exceeds 12 km, so that the density variation due to compression is quite

small. In the atmosphere, however, $c \approx 300$ m/s and $H \approx 10$ km. Thus the exponential decrease of density with height is an important consideration.

The last result predicts that density variations in the ocean due to compression are about 1%. The observed density variations at atmospheric pressure due to temperature and dissolved salt content are about the same order of magnitude. However, it is not difficult to argue that the density variation due to compression is not dynamically significant. Consider the two small fluid parcels shown in Figure 10. Suppose that they have identical densities at atmospheric pressure. Because of the compression the deeper parcel will have a higher actual density than the shallower parcel. Normally, one would expect to have to do work against the gravitational force in order to interchange these two parcels because the lighter one is on top. But what actually happens? After the interchange, compression of the parcel that goes down is matched by the expansion of the parcel that goes up. Thus the density structure after the interchange is the same as it was before. The gravitational potential energy of the situation is unchanged and hence no work was necessary.

More generally we want to know whether a given density distribution will release or absorb gravitational potential energy if it is perturbed (i.e. whether it is unstable or stable). It should be clear from the above simple discussion that it is not the *in situ* density that is important, but the density of the fluid after decompression to some standard pressure (usually one atmosphere). You should note that any such test is sufficient to prove stability, but not instability. This is because any real interchange involves viscous dissipation. Thus instability always requires that enough gravitational potential energy be released to overcome this friction.

One additional complication must also be considered. When a fluid parcel is decompressed without allowing heat to flow in or out of the parcel, its temperature will drop. This is called the adiabatic temperature change. A rigorous consideration of the interchange of the two parcels considered earlier concludes that the correct test for stability requires that the parcels be interchanged adiabatically. This means that they must be decompressed to the standard pressure in an insulated bottle before their densities are compared. This adiabatically decompressed density is called the potential density. The adiabatic temperature gradient is quite small (about 0.11 ^{o}C per km; see Tritton section 14.6) in the ocean and not a very important correction unless one is considering circulation from the surface to the bottom of the deep trenches (which exceed 10 km). However, it is critical in considering the stability of the atmosphere, stars and Planetary interiors.

Fortunately, appropriate measurements of density are natural to the way ocean samples were collected historically. Water was brought to the surface in a mechanical bottle which had a thermometer mounted on its side that could be mechanically triggered to record the temperature at the sampling depth. Density was routinely measured at this *in situ* temperature and atmospheric pressure. If more accuracy was desired, the density was measured at the potential temperature (the *in situ* temperature corrected for the adiabatic temperature change) and atmospheric pressure. This density is the potential density.

Modern techniques typically involve measuring the electrical conductivity and temperature *in situ* with an instrument package called an STD or CTD lowered on a cable. The electrical conductivity is directly related to the salinity (S) and temperature (T) and

the relation between S, T and ρ at atmospheric pressure has been determined empirically to high accuracy. Salinity is the number of grams of dissolved non-organic solids per kilogram of water (usually written $^o/_{oo}$) with some caveats such as the fact that most methods can not distinguish between chlorine, iodine and bromine ions and so assume they are all chlorine. Salinity is useful because the relative proportions of cations and anions in ocean water is constant throughout the oceans except near the mouths of rivers and the vents of ocean bottom hydrothermal vents. This presumably reflects the high degree of mixing over geologic time.

The density variations in the ocean are so small that oceanographers usually quote the quantity

$$\sigma = (\rho - 1) \times 10^3$$

instead of the density. They use σ_T for this quantity evaluated at *in situ* temperature and atmospheric pressure and σ_* instead of the potential density.

3.3. Couette Flow

I have already describe the geometry of Couette flow (Figure 6). Couette flow has four important features. First, it is steady and therefore $\frac{\partial \mathbf{u}}{\partial t} = 0$. Second, $\mathbf{u}$ varies spatially only perpendicular to its direction. For the geometry of Figure 6, $\mathbf{u} = u(z)\hat{\mathbf{x}}$. Thus $(\mathbf{u} \cdot \nabla)\mathbf{u} = u\frac{\partial u}{\partial x} = 0$. Third, it is horizontal so that gravity has no component in the direction of flow and hence no importance other than to establish the hydrostatic pressure. And finally, because the boundary moves tangential to itself, it exerts no pressure on the fluid and hence $p' = 0$. Thus the conservation of momentum reduces to the simple ordinary differential equation

$$\frac{d^2 u}{dz^2} = 0$$

with general solution

$$u(z) = Bz + C$$

If we let the bottom boundary be at z = 0 and let the u(0) = 0 (i.e. consider the bottom fixed), we immediately conclude that C = 0. If the top boundary is at z = D and its velocity is U, $B = \frac{U}{D}$. and

$$u(z) = \frac{U}{D} z$$

If we wish to specify the shear stress at the top as T_{xz} instead of the velocity, we must have

$$\left[\frac{du}{dz}\right]_D = B = \frac{T_{xz}}{\mu}$$

and

$$u(z) = \frac{T_{xz}}{\mu} z$$

Equating the two solutions, we conclude that

$$\mu = \frac{T_{xz} D}{U}$$

Since the strain rate in the fluid, $\frac{\partial u}{\partial z} = B$ is constant at all values of z, one can in principle use the last result to determine an effective dynamic viscosity from measurements of T_{xz} and U even for a non-Newtonian fluid. In practice, however, measuring the tangential force per unit area on a large horizontal plate moving over a layer of fluid is not very convenient. Thus Couette flow is realized experimentally in other ways. Two are illustrated in Figure 11. The first is the flow between concentric cylinders which rotate relative to one another and the second is the flow between a plate and a rotating cone whose apex just touches the plate. There are two basic constraints that must be satisfied for these to be good approximations of Couette flow. The first is purely geometrical. For the concentric cylinders, the ratio of the cylinder spacing δ to the radius of the outer cylinder R must be large. The second is physical. Since rotation is involved, the Coriolis force in the fluid must be everywhere small compared to the viscous force. For a patch of the flow of area A between the concentric cylinders, we want

$$\mu \,|\frac{\partial u}{\partial R}|\, A \gg \rho \,|2\Omega \times \mathbf{u}|\, A \, \delta$$

We can estimate

$$|\frac{\partial u}{\partial R}| \approx \frac{U}{\delta}$$

and

$$|2\Omega \times \mathbf{u}| \approx 2\Omega U$$

where U is the velocity difference between the inner and outer boundaries. If the inner cylinder is stationary, $U = \Omega R$. If both cylinders rotate at rates that differ by $\Delta\Omega$, we can take $U = \Delta\Omega \, R$. Our actual estimate for U is not critical because both U and A cancel out of the required condition and we get

$$\mu \frac{1}{\delta} \gg \rho 2\Omega\delta$$

or

$$\delta \ll \sqrt{\frac{v}{2\Omega}}$$

The length scale on the right is called the Ekman thickness and will be very important in another context later. For $\Omega = 0.5$ rpm ($\frac{\pi}{60}$ radians per second) and $v = 0.01 \ cm^2/s$ (water), $\delta \ll 3$ mm. For an automobile motor oil ($v \approx 1 \ cm^2/s$), $\delta \ll 3$ cm. When the conditions for Couette flow fail, the flow exhibits interesting instabilities that have been extensively studied (see Tritton Section 17.5). One can go through a similar argument for the cone-plate configuration, and conclude that the angle between the cone and the plate must always be very small.

3.4. Poiseuille Flow

3.4.1. Plane geometry

This flow differs from plane horizontal Couette flow in that both boundaries are fixed and the flow is driven by a constant negative pressure gradient -G in the direction of flow. Thus the Navier-Stokes equation becomes

$$\frac{d^2u}{dz^2} = \frac{G}{\mu}$$

which can easily be integrated to give

$$u(z) = \frac{G}{2\mu} z^2 + Bz + C$$

Letting u = 0 at z = 0 and D, this becomes

$$u(z) = \frac{G}{2\mu} (z^2 - Dz)$$

This velocity has a parabolic form with the maximum velocity at the plane of symmetry at $\frac{D}{2}$ (see Figure 12(a)).

3.4.2. Pipe flow

Poiseuille flow also occurs in a pipe of circular cross-section. It is appropriate to use cylindrical coordinates aligned with the pipe (see Figure 12(b)). The velocity field will have the general form $\mathbf{w} = w(R)\hat{\mathbf{z}}$. Note that this $\hat{\mathbf{z}}$ is along the pipe axis and *not* the direction of gravity. If gravity has any component along the pipe (i.e the pipe is not horizontal) the associated hydrostatic pressure gradient will add to (or subtract from) the pressure gradient driving the flow. Otherwise the arguments regarding the neglect of terms in the Navier-Stokes equation are identical to plane Poiseuille and Couette flow and

the equation becomes

$$\frac{1}{R}\frac{d}{dR}(R\frac{dw}{dR}) = \frac{G}{\mu}$$

Multiplying this equation by R and integrating once gives

$$R\frac{dw}{dR} = \frac{G}{2\mu}R^2 + B$$

Then dividing by R and integrating again gives

$$w(R) = \frac{G}{4\mu}R^2 + B\ln R + C$$

We must take B = 0 in order to avoid an infinite singularity on the axis of the pipe. Such a singularity would imply infinite kinetic energy in any finite length of the pipe and is clearly physically impossible. The remaining constant is determined by the requirement that the velocity go to zero at R = a, the inner radius of the pipe and we finally obtain

$$w(R) = \frac{G}{4\mu}(a^2 - R^2)$$

The maximum velocity is on the pipe axis and is $w(0) = \dfrac{Gd^2}{4\mu}$. We can then rewrite the velocity field in the non-dimensional form

$$\frac{w(R)}{w(0)} = 1 - (\frac{R}{a})^2$$

This clearly demonstrates that the shape of the velocity is independent of the radius of the pipe, the material properties or the driving pressure gradient and that it always has the same parabolic form as the plane flow, but is cylindrically symmetric about the axis of the pipe. We call such a family of solutions dynamically similar.

Integrating cylindrical Poiseuille flow over the cross-section of the pipe gives the total volume flux

$$F = 2\pi w(0)\int_0^a [1 - (\frac{R}{a})^2]\ RdR$$

$$= \frac{\pi}{2}w^2(0) = \frac{\pi Ga^4}{8\mu} = \frac{A}{2}w(0)$$

where A is the cross-sectional the area of the pipe. Note the very strong dependence of

the flux on a. The average fluid velocity

$$w_{avg} = \frac{F}{A} = \frac{1}{2}\,w(0) = \frac{a^2}{8\mu}\,G$$

is directly proportional to the magnitude of the pressure gradient.

This result can be re-arranged to give

$$\mu = \frac{a^2 G}{8 w_{avg}}$$

Thus pipe flow can also be used to measure the dynamic viscosity of a fluid. A pressure difference that is commonly used to establish an accurate pressure gradient is between the atmosphere and a vacuum. Note, however, that the shear stress on any cylindrical surface in the flow depends on the radial position. It is zero at the center of the pipe and maximum at the wall. Thus the flow profile will be parabolic only for a Newtonian fluid and a pipe viscometer is not very convenient for non-Newtonian fluids.

Finally, Poiseuille flow (and Couette flow) is not a unique solution to the full Navier-Stokes equation. It exists only when the other terms can be guaranteed to be small. For instance, if the flow deviates from being parallel to the pipe wall, the advective term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is no longer identically zero. To be certain that we can ignore this term we require that

$$(\mathbf{u} \cdot \nabla)\mathbf{u} \ll \nu\nabla^2\mathbf{u}$$

If we estimate the magnitude of $\mathbf{u}$ as the velocity at the center of the pipe ($w(0) = W$) and the magnitude of the velocity gradients by $\dfrac{W}{a}$, the condition for neglect of the advective term becomes

$$1 \gg \frac{Wa}{\nu} \equiv R_e$$

which is called the Reynolds Number. Note that this number is non-dimensional. When R_e is small, viscous effects dominate the flow dynamics. When it is large, viscous effects are presumably much less important.

Experimentally, it is observed that laminar (i.e stable) Poiseuille flow breaks down when R_e exceeds about 1000 (a much higher value than one might expect from our discussion above, which suggests that a value of order 1 might more likely). The exact value depends on the smoothness of the inside of the pipe and the upstream conditions (i.e. how the fluid gets into the pipe).

The breakdown of laminar pipe flow is of great interest to engineering because the additional kinetic energy in the eddies and oscillations of the unstable flow must be

supplied by the mean flow. This implies that the pressure gradient required to drive the flow will be higher than one would expect for the viscosity of the fluid. At very high Reynolds number the flow becomes completely chaotic (i.e. fully turbulent) and if $\bar{\mathbf{u}}$ is defined as the velocity in a finite volume averaged over a time scale long compared to the fluctuations of the flow, the radial profile of $\bar{\mathbf{u}}$ is very similar to Poiseuille flow. Furthermore, like Poiseuille flow, the average velocity through the pipe is linearly related to the driving pressure gradient (see Tritton Figure 2.11). If one did not know that the flow was turbulent, one could calculate a dynamic viscosity from the average velocity and the pressure gradient which would turn out to be much higher than the molecular value for the fluid. It is is called the eddy viscosity ν_{eddy} because momentum (i.e $\rho\bar{\mathbf{u}}$) is transferred from fast regions of the flow to slow regions by mixing rather than by the collisions responsible for molecular viscosity. Since eddies must involve local circulation, the correlation between the radial and longitudinal velocity components (i.e. $\overline{uw}$) is related to the strength of the mixing. This cross correlation is called the Reynolds stress and it plays the same role for turbulent flow that shear stress does for laminar flow. Thus

$$ -\overline{uw} = \nu_{eddy}\frac{\partial \bar{w}}{\partial R} $$

(see Tritton section 19.3 for further discussion). The difference in the velocity profile between laminar and fully turbulent Poiseuille flow can be attributed to the fact that the size spectrum of the mixing eddies is restricted close to a wall and hence ν_{eddy} is smaller near the wall. Geophysical flows are almost all turbulent to some extent. However by adopting the eddy viscosity model for approximating turbulence we can understand many phenomena with the equations appropriate to laminar flow.

3.4.3. Porous media

Many important geophysical materials (sediments, soils, rocks) contain small interconnected pores. Empirically, it is observed that the volume flux per unit area u_{avg} through a porous sample (see Figure 13(a)) often obeys the relation

$$ u_{avg} = \frac{k}{\mu}G $$

where again -G is the pressure gradient, μ is the dynamic viscosity of the fluid in the pores and k is a constant called the permeability (not to be confused with a magnetic quantity with the same name). This relationship is called Darcy's Law. The linear relationship between w_{avg} and G for pipe flow suggests that Darcy's Law is exactly what we should expect if the Reynolds Number of the flow in the pores is small enough that we have Poiseuille flow.

Two-dimensional flow in a porous medium can be visualized in an apparatus called a Hele-Shaw cell. This consists of two transparent parallel plates separated by a fluid filled gap of thickness D (see Figure 13(b)). Integrating the solution for plane Poiseuille flow across the gap we can conclude that

$$u'_{avg} = \frac{1}{D} \int_0^D \frac{G}{2\mu} (Dz - z_2) \, dz = \frac{D^2}{12\mu} G$$

If u'_{avg} is identified with u_{avg} for a porous medium, we can conclude that the permeability for a Hele-Shaw cell is

$$k = \frac{D^2}{12}$$

and you will see this value widely used in the literature. However, Darcy defined u_{avg} as the mean flux per unit area through the the entire area of the face of a porous medium including the material between the pores. To be consistent with this definition, the permeability of a cell whose total width (walls plus gap) is H should be

$$k = \frac{D^3}{12H}$$

This may seem like a trivial point and it is not consequential for situations where the properties of the walls are unimportant. However, there are several papers in the literature using Hele-Shaw cells to study thermal convection in a porous medium which are wrong because the authors failed to understand that use of the first definition requires redefining the heat transport by the walls in a way which prevents direct use of results derived for an ordinary porous medium. The second definition avoids the problem entirely because it is completely consistent with Darcy's definition of permeability.