6. Surface Waves

Waves are the inevitable physical consequence of forces that try to restore equilibrium after a perturbation. They are tremendously important in the environment, because they are an efficient way for mechanical energy to be transported from one part of the system to another. In addition their consequences are often quite easy to see. Furthermore there are many similarities between waves that have restoring forces arising from very different physics. Before discussing specific examples, I will review several concepts common to the analysis of all small amplitude waves.

6.1. Perturbation expansions

In previous chapters, the Navier-Stokes equations became linear because the advection term was small or identically zero for reasons related to the flow geometry or scale. This non-linear term can also be negligible when the amplitude is small. Such situations are often best handled using a perturbation expansion. We assume that we know a solution to the relevant equations that is close to the situation we are actually interested in. We can then solve the problem by finding the difference between the known and desired solutions. This difference will obey a simpler linear partial differential equation.

To illustrate the technique while keeping the algebra to the absolute minimum, I will first consider the conservation of mass (continuity) equation for the case of an incompressible fluid

$$\nabla \cdot \mathbf{u} = 0$$

Let

$$\mathbf{u} = \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \varepsilon^3 \mathbf{u}_3 + \cdots$$

$$p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \varepsilon^3 p_3 + \cdots$$

where $\mathbf{u}_0(\mathbf{x}, t)$ and $p_0(\mathbf{x}, t)$ are the solution we know and $\varepsilon \mathbf{u}_1$, $\varepsilon^2 \mathbf{u}_2$, εp_1, $\varepsilon^2 p_2$ etc. are successively smaller corrections to the initial solution. The constant $\varepsilon < 1$ serves two purposes. First, it allows us to easily see what happens if we adjust the size of the perturbation. Second, it serves as a convenient book-keeping tool since we expect that terms involving successively higher powers of ε to get smaller and smaller. Substituting the expansion for $\mathbf{u}$ into the continuity equation gives

$$\nabla \cdot (\mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \ldots) = \nabla \cdot \mathbf{u}_0 + \varepsilon \nabla \cdot \mathbf{u}_1 + \varepsilon^2 \nabla \cdot \mathbf{u}_2 + \cdots = 0$$

Now we have assumed that $\mathbf{u}_0$ already satisfies the *zeroth order equation*

$$\nabla \cdot \mathbf{u}_0 = 0$$

Subtracting this equation from the expanded equation and dividing by ε gives

$$\nabla \cdot \mathbf{u}_1 + \varepsilon \nabla \cdot u_2 + \cdots = 0$$

As ε becomes small, this equation reduces to the *first order equation*

$$\nabla \cdot \mathbf{u}_1 = 0$$

which (with the other first order equations and boundary conditions) can be solved for $\mathbf{u}_1$. Once $\mathbf{u}_1$ is known, we can subtract this last equation from the previous one, divide by epsilon and derive the *second order equation*

$$\nabla \cdot \mathbf{u}_2 = 0$$

This process can be continued for as many orders as one desires. The higher the order, the better the accuracy. Alternatively, the higher the order, the larger ε can be for a specified accuracy.

The continuity equation is linear so that the expansion process decouples the orders and each order satisfies the same equation. This is not true for non-linear terms such as the advection in the Navier-Stokes equation. Substituting the velocity expansion into this term gives

$$\mathbf{u} \cdot \nabla \mathbf{u} = (\mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \ldots) \cdot \nabla(\mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \ldots)$$

$$= \mathbf{u}_0 \cdot \nabla \mathbf{u}_0$$

$$+ \varepsilon(\mathbf{u}_0 \cdot \nabla \mathbf{u}_1 + \mathbf{u}_1 \cdot \nabla \mathbf{u}_0)$$

$$+ \varepsilon^2(\mathbf{u}_0 \cdot \nabla \mathbf{u}_2 + \mathbf{u}_2 \cdot \nabla \mathbf{u}_0 + \mathbf{u}_1 \cdot \nabla \mathbf{u}_1)$$

$$+ \varepsilon^3(\mathbf{u}_0 \cdot \nabla \mathbf{u}_3 + \mathbf{u}_3 \cdot \nabla \mathbf{u}_0 + \mathbf{u}_2 \cdot \nabla \mathbf{u}_1 + \mathbf{u}_1 \cdot \nabla \mathbf{u}_2)$$

Note that the first order term (the one multiplied by ε) is linear in $\mathbf{u}_1$, the second order term is linear in $\mathbf{u}_2$ and the third order term is linear in $\mathbf{u}_3$. In general, the zeroth order term is not linear, but we assume from the start that we know a solution to the zeroth order equation.

Consider the very important example of a perturbation to the state of rest $\mathbf{u}_0 = 0$, ρ constant. The zeroth order ($\varepsilon = 0$) Navier-Stokes equation becomes

$$\frac{1}{\rho} \nabla p_0 = -g\hat{\mathbf{z}}$$

and as you might expect we simply have hydrostatic pressure. Subtracting this equation from the full (expanded) Navier-Stokes equation, dividing by ε and then letting $\varepsilon \to 0$ gives the first order equation

$$\frac{\partial \mathbf{u}_1}{\partial t} + 2\Omega \times \mathbf{u}_1 + \frac{1}{\rho}\nabla p_1 = \nu \nabla^2 \mathbf{u}_1$$

which is independent of gravity because we have assumed ρ constant. This linear equation can be solved for $\mathbf{u}_1$ and then the second order equation is

$$\frac{\partial \mathbf{u}_2}{\partial t} + \mathbf{u}_1 \cdot \nabla \mathbf{u}_1 + 2\Omega \times \mathbf{u}_2 + \frac{1}{\rho}\nabla p_2 = \nu \nabla^2 \mathbf{u}_2$$

This equation is again linear because $\mathbf{u}_1$ is already known.

6.2. Gravity waves

I will now consider waves on the surface of a constant density ocean. These are not covered in Tritton, although he discusses several other types of waves and the concepts of phase and group velocity.

The ratio of the magnitudes of the acceleration to Coriolis terms in the first order Navier-Stokes equations is

$$\frac{|2\Omega \times \mathbf{u}_1|}{\left|\frac{\partial \mathbf{u}_1}{\partial t}\right|} \approx 2\Omega\tau$$

If you have been to an ocean beach, you know that surface waves are dominated by periods τ less than 100 seconds Thus $2\Omega\tau <\approx 10^{-2}$ and the Coriolis force is negligible. The first order equations for surface waves are therefore

$$\frac{\partial \mathbf{u}_1}{\partial t} + \frac{1}{\rho}\nabla p_1 = \nu \nabla^2 \mathbf{u}_1$$

and

$$\nabla \cdot \mathbf{u}_1 = 0$$

Taking the divergence of the first equation and using the second immediately implies

$$\nabla^2 p_1 = 0$$

Thus the first order perturbation pressure satisfies the Laplace equation. Alternatively, taking the curl of the first equation immediately implies that

$$\frac{\partial}{\partial t}\nabla \times \mathbf{u}_1 = \frac{\partial \omega_1}{\partial t} = 0$$

Thus the first order vorticity of the motion must be constant in time. The only constant vorticity consistent with a motion that oscillates around a state of rest is zero. Thus $\mathbf{u}_1$ is irrotational and can be represented as $\mathbf{u}_1 = \nabla\phi_1$. The vanishing of the divergence of $\mathbf{u}_1$ then implies that

$$\nabla^2\phi_1 = 0$$

This equation further implies that

$$\nabla^2\mathbf{u}_1 = \nabla^2(\nabla\phi_1) = \nabla(\nabla^2\phi_1) = 0$$

and hence that the viscous term is zero in the first order equations. This is why small amplitude surface gravity wave energy can propagate across entire ocean basins.

To complete the statement of the problem we need to consider the boundary conditions. The $\hat{\mathbf{z}}$ component of the first order Navier-Stokes equation is

$$\frac{\partial w_1}{\partial t} + \frac{1}{\rho}\frac{\partial p_1}{\partial z} = \nu\nabla^2 w_1$$

Where I have retained the viscous term for the moment. The obvious condition on the vertical velocity at the ocean floor is $w_1 = 0$ (unless it is extremely porous) and therefore this equation implies that the boundary condition on the pressure at the bottom must be

$$@\ z = -H: \qquad \frac{\partial p_1}{\partial z} = 0$$

If the top surface is perturbed vertically a distance h, the pressure change will be $p_1 = \rho gh$ (see Figure 31). Taking the time derivative of this gives

$$\frac{\partial p_1}{\partial t} = \rho g\frac{\partial h}{\partial t} = \rho g w_1$$

This relation can be used to eliminate w_1 in the $\hat{\mathbf{z}}$ component of the Navier-Stokes equation to give

$$@\ z = 0: \qquad \frac{\partial^2 p_1}{\partial t^2} + g\frac{\partial p_1}{\partial z} = \nu\frac{\partial}{\partial t}(\nabla^2 p_1)$$

However, because p_1 satisfies the Laplace equation, this immediately simplifies to

$$@\ z = 0: \qquad \frac{\partial^2 p_1}{\partial t^2} + g\frac{\partial p_1}{\partial z} = 0$$

This relation is called a *kinematic boundary condition* and is in fact the only way that time enters the problem. Note that again viscosity completely drops out of the problem to first order.

The condition required to complete the problem is the initial shape of the top surface. A basic theorem due to Fourier states that any arbitrary height variation can be represented as a sum of cosine and sine waves. As long as the equations do not involve products of the solution (i.e. the wave amplitude is small enough that the first order equation gives sufficient accuracy), all the individual waves in the sum are independent of each other. Furthermore the only difference between a cosine and a sine is a horizontal shift of one quarter wavelength (see Figure 32(a)). Thus it is sufficient to solve the problem for a single cosine perturbation of arbitrary wavelength. We shall therefore consider the time history of an initial two-dimensional surface height perturbation

$$h(x) = A \cos(kx) = A\,\mathrm{Re}[e^{ikx}]$$

where $k = \dfrac{2\pi}{\lambda}$ is known as the wavenumber, λ is the wavelength and Re[ ] means ‘‘the real part of’’. The wavenumber is the spatial frequency and has the units of radians per meter. Because $p = \rho g h$ at $z = 0$, the above condition obviously implies that

$$@z = 0;\ t = 0 \qquad p_1 = \rho g A\,\mathrm{Re}[e^{ikx}]$$

Since we have assumed a two-dimensional initial height perturbation, the subsequent motion will also be two-dimensional and be restricted to the x-z plane. The first order pressure must then satisfy the two-dimensional Laplace equation

$$\frac{\partial^2 p_1}{\partial x^2} + \frac{\partial^2 p_1}{\partial z^2} = 0$$

This equation is separable in rectangular Cartesian coordinates which means that it can be written as function which depends only on x times a function which depends only on z. (This technique was discussed in Chapter 4 and is covered in many texts on electrostatics or potential theory.) It is then easy to show that

$$p_1 = [F(t)e^{+kz} + G(t)e^{-kz}]\ e^{ikx}$$

satisfies both the Laplace equation and the x-dependence of the initial surface pressure. The two functions F(t) and G(t) are the integration ‘‘constants’’ of the solution. They can depend on t because the Laplace equation is independent of t.

6.2.1. Deep water waves

At this point, we will further simplify the problem by assuming that the water is deep enough that $H \rightarrow \mathit{infinity}$. The boundary condition at the bottom then obviously requires that G(t) = 0. Substituting the solution into the kinematic boundary condition gives

$$\frac{\partial^2 F}{\partial t^2} + gkF = 0$$

which has a solution

$$F(t) = C\ \cos\,(\sigma t)$$

where

$$\sigma = \pm\sqrt{gk}$$

The constant C is determined so as to match the amplitude of the initial pressure variation at z=0. Pulling everything together, we find

$$p_1(x, z, t) = \rho g A\ e^{kz}\ Re[e^{i(kx\,\pm\,\sigma t)}] = \rho g A\ e^{kz}\ \cos(kx \pm \sigma t)$$

6.2.2. Phase and group velocity

The argument of the cosine function is called its phase. The phase will be constant when

$$kx \pm \sigma t = constant$$

An observer who moves at the so-called phase velocity

$$c_p \equiv \pm\frac{\sigma}{k}$$

will see a constant phase (i.e. wave height). Thus the phase velocity is the velocity of the crests and troughs of the wave. The negative sign of σ is a wave moving to the right (i.e. towards +x) while the positive sign is a wave moving to the left. In the remainder of this discussion we will concentrate on the case with the negative sign.

The phase velocity is not the full story regarding the propagation of a wave. Suppose you have two waves $h_1 = \frac{A}{2}\sin(k_1 x - \sigma_1 t)$ and $h_2 = \frac{A}{2}\sin(k_2 x - \sigma_2 t)$ with equal amplitudes, but different spatial and temporal frequencies. Adding these two waves together and using the standard trigonometric addition formula

$$\cos(\alpha) + \cos(\beta) = 2\cos(\frac{\alpha - \beta}{2})\cos(\frac{\alpha + \beta}{2})$$

gives

$$h_1 + h_2 = A\ \sin[\frac{k_1 - k_2}{2}\ x - \frac{\sigma_1 - \sigma_2}{2}\ t]\ \sin[\frac{k_1 + k_2}{2}\ x - \frac{\sigma_1 + \sigma_2}{2}\ t]$$

The combined wave looks like either of the original waves multiplied by a slowly varying envelope (see Figure 32(b)). You may be familiar with this phenomenon from the ‘‘beating’’ of two tuning forks with almost identical frequency. To remain at a fixed amplitude of the envelope, you need to move at the velocity

$$c_g = \frac{\sigma_1 - \sigma_2}{k_1 - k_2}$$

As the frequencies and wavenumbers of the two waves approach each other this becomes the so-called group velocity

$$c_g = \frac{\partial \sigma}{\partial k}$$

When a spectrum of waves are present it is possible to have an envelope that is zero outside some finite region which is called a *wave group*. The wave energy is non-zero only where the envelope is non-zero, i.e. inside the wave group. Thus the energy of the waves travels at the envelope or group velocity. An interesting point is that a single sine wave carries no information other than its frequency. To transmit information, it is necessary to have more than one frequency. If you mix waves so as to create an envelope that is not constant you have amplitude modulation (AM, commonly used for low frequency radio transmission). If you mix waves so that the frequency fluctuates, you have frequency modulation (FM, commonly used for high frequency radio and television transmission).

For deep water surface waves

$$c_p = \frac{\sigma}{k} = \sqrt{\frac{g}{k}} = \frac{g}{\sigma}$$

and

$$c_g = \frac{\partial}{\partial k}\sqrt{gk} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}c_p$$

Both the phase and group velocities increase with increasing wavelength (decreasing k).

The relationship between σ and k is called the dispersion relation, because it predicts that waves of different wavelength or frequency that are initially at the same place in space will move at different velocities and thus subsequently disperse. If we have a patch of ocean that initially contains a spectrum of wavelengths (due to a storm, for instance), the patch with non-zero envelope (called the wave group) containing this energy will change with time as the group propagates. The short wave energy will lag behind the long wave energy and will be found near the back of the group (see Figure 33). The long wave energy will travel fastest and will be found near the front of the group. As the group propagates, it will stretch (disperse) with time. We also know that the velocity of the crests and troughs is twice that of the envelope. Thus crests and troughs move faster than the envelope and will move through the group. An individual pair of crests comes in the back of the group with small amplitude and close spacing (short wavelength). Their amplitude grows and their wavelength increases as they progress through the group. Finally their amplitude dies and their separation is greatest as they pass out of the front of the group (see Figure 33). Consequently, a wave crest has a finite lifetime and you cannot surf indefinitely on an individual crest! It is wave groups that propagate across oceans and not wave crests. If you observe the arrival of the group at a beach, you will see the long wave (low frequency) energy arrive first. By measuring the lag time between the arrival of the dominant wave energy at different frequencies (see Figure 34), you can estimate how far back in time (and hence space) you have to go before wave energy of all frequencies was coincident in space (i.e. how far away the storm was).

6.2.3. Velocities, particle orbits and linearity

Several other points can be made regarding deep water gravity waves. To begin with, substituting the first order pressure into the $\hat{\mathbf{z}}$ component of the Navier-Stokes equation we have

$$\frac{\partial w_1}{\partial t} = \frac{-1}{\rho}\frac{\partial}{\partial z}[\rho g A e^{kz}\cos(kx-\sigma t)] = -\,gkAe^{kz}\cos(kx-\sigma t)$$

Integrating with respect to time and then using the dispersion relation results in the first order vertical velocity

$$w_1 = \frac{gkA}{\sigma}\, e^{kz}\,\sin(kx-\sigma t) = \sigma A\, e^{kz}\,\sin(kx-\sigma t)$$

The integration constant is zero because the vertical velocity of the water surface time-averaged over a complete cycle must be zero.

The first order horizontal velocity can be obtained by substituting p_1 into the $\hat{\mathbf{x}}$ component of the Navier-Stokes equation and proceeding as above or, alternatively, substituting w_1 into the two-dimensional continuity equation

$$\frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z} = 0$$

and integrating with respect to x giving

$$u_1 = \sigma A\, e^{kz}\,\cos(kx-\sigma t)$$

Integrating the velocities with respect to time gives the particle displacements

$$x_1 = -\,A\, e^{kz}\,\sin(kx-\sigma t)$$

$$z_1 = A\, e^{kz}\,\cos(kx-\sigma t)$$

Again, the integration constants vanish because the time-averaged top surface is at z=0. These displacements represent circular particle motion which is in the direction of the phase velocity at the top of the circle (see Figure 35(a)) and is therefore called prograde motion.

The radius of the circle decreases with depth and will be only 5% of its surface value at a depth $z = \frac{3}{k} \approx \frac{\lambda}{2}$. Thus the deep water solution should be an excellent approximation in water deeper than half a wavelength. Swell seen in the open ocean with a period of 10 seconds has $\sigma = \frac{2\pi}{10} = 0.31\ sec^{-13}$; $k = \frac{\sigma^2}{g} = 0.040\ m^{-1}$; and $\lambda = \frac{2\pi}{k} = \frac{2\pi g}{\sigma^2} = 160$ meters. Consequently, to be deep water waves, swell requires water of order 80 meters or more depth. Since water of this depth typically involves going well out on the continental shelves, the longer period ocean waves that one sees near shore are almost always significantly affected by finite water depth. Much shorter period waves (with periods as short as 1 second) can be seen in areas like Puget Sound or Lake Washington. A 1 second wave has a wavenumber of 4 m^{-1} and a wavelength of only 1.6 meters and is thus much more likely to be a deep water wave near shore.

The legitimacy of the first order linearization depends on $\mathbf{u}_1 \cdot \nabla \mathbf{u}_1$ being much less than $\frac{\partial \mathbf{u}_1}{\partial t}$. We must therefore have

$$\frac{|\mathbf{u}_1 \cdot \nabla \mathbf{u}_1|}{\left|\frac{\partial \mathbf{u}_1}{\partial t}\right|} \approx \frac{k|\mathbf{u}_1|^2}{\sigma|\mathbf{u}_1|} = \frac{|\mathbf{u}_1|}{\sigma/k} = \frac{|\mathbf{u}_1|}{c_p} \ll 1$$

From the results derived above, $|\mathbf{u}_1| \approx \sigma A$. Thus an alternate form of the linearity condition is

$$\frac{|\mathbf{u}_1|}{\sigma/k} = Ak = \frac{2\pi A}{\lambda} \ll 1$$

which is just the $(Strouhal\ number)^{-1}$ evaluated for $U \approx \sigma A$, $T \approx \frac{1}{\sigma}$ and $L \approx \frac{1}{k}$. For swell with a 10 second period, $\lambda = 160$ meters, and linearity requires that the crest to trough height (2A) be substantially smaller than 50 meters. It is easy to show that when the linearity condition condition fails (i.e. $A = \frac{1}{k}$), the surface acceleration of the wave equals gravity. Obviously this wave amplitude cannot be exceeded under any circumstances since gravity is the wave's restoring force*. Waves with crest to trough height exceeding 20 meters are extremely rare in the open ocean even in the roughest conditions. Thus one never approaches this extreme limit. However, for 1 second waves the critical

*It should be noted that seismic surface waves *can* have surface accelerations that exceed gravity because energy is stored in the elasticity of the material in addition to the gravitational potential energy.

half-amplitude is only 0.25 meters and non-linear effects such as asymmetry in the leading and trailing sides of wave crests (causing the waves to lean forward in their direction of propagation), smaller curvature in troughs than at crests and changes in the frequency spectrum as waves of different wavenumbers and periods interact are commonly seen. One can study these effects by going to higher order approximations.

The wind is actually most efficient at generating very short period waves on the ocean surface. It is the importance of the non-linear effects for these shorter waves that conspires in areas of active wave generation to cause short wavelength energy to be converted into longer wavelength energy or to be dissipated in turbulence (white capping). The ‘‘cascading’’ of energy from the short waves to the long waves continues until the waves become long enough that the non-linear effects weaken. This is why the wave energy in the open ocean has a spectral peak at a period of order 10 to 30 seconds (see Figure 34). Detailed consideration of these processes is beyond the scope of this course.

We can also use the calculated scale for the velocity, $U \approx \sigma A$ to estimate the size of other terms in the momentum equation relative to the time-dependent inertia.

$$\frac{|\Omega \times \mathbf{u}_1|}{\left|\frac{\partial \mathbf{u}_1}{\partial t}\right|} \approx \frac{2\Omega\sigma A}{\sigma(\sigma A)} = \frac{2\Omega}{\sigma} = \frac{1}{Strouhal\ number \times Rossby\ number}$$

This is a re-statement of our justification for ignoring the Coriolis term as long as the wave period is small compared to a day.

$$\frac{|\nu\nabla^2\mathbf{u}_1|}{\left|\frac{\partial \mathbf{u}_1}{\partial t}\right|} = \frac{\nu\sigma A k^2}{\sigma(\sigma A)} = \frac{\tau^{-1}}{\sigma} = \frac{1}{Strouhal\ number \times Reynolds\ number}$$

where

$$\tau = \frac{1}{k^2\nu} = \frac{L^2}{\nu}$$

is the times scale for diffusion of a disturbance of length scale $L = \frac{1}{k}$ in a medium of viscosity ν. Since this diffusion time is always very much longer than the wave period, we would be justified in ignoring viscous damping even if the wave solution did not make the viscous term identically zero.

6.2.4. Shallow water waves

If the water is much shallower than half a wavelength there will be a major difference in the solution. The vertical fluid displacement and velocity must go to zero at the bottom (unless it is very porous) and hence the scale for vertical variations must be the water depth H rather than $\frac{1}{k}$ of the exponential variation in the deep water solution. There are two ways to approach this problem.

The first is to consider the effect of finite water depth on the deep water solution. Keeping both the +kz and -kz exponential terms in the solution to the Laplace equation and applying the boundary condition

$$@z = -H: \qquad \frac{\partial p_1}{\partial z} = 0$$

leads to the dispersion relation

$$\sigma = \sqrt{gk \tanh(kH)}$$

and the solution

$$p_1 = \rho g A \frac{\cosh k(z+H)}{\cosh kH} cos(kx - \sigma t)$$

$$w_1 = \sigma A \frac{\sinh k(z+H)}{\sinh kH} \sin(kx - \sigma t)$$

$$. sp0.5 z_1 = A \frac{\sinh k(z+H)}{\sinh kH} cos(kx - \sigma t)$$

$$u_1 = \sigma A \frac{\cosh k(z+H)}{\sinh kH} cos(kx - \sigma t)$$

$$x_1 = -A \frac{\cosh k(z+H)}{\sinh kH} sin(kx - \sigma t)$$

We can then use the following asymptotic relations for the hyperbolic functions:

$$x \to infinity: \qquad sinh(x) = cosh(x) \to \frac{e^x}{2} \qquad tanh(x) \to 1$$

$$x \to 0: \qquad sinh(x) = tanh(x) \to x \qquad cosh(x) \to 1$$

to conclude that, when $H \gg \frac{1}{k}$, the above solution reduces to the deep water solution and when H becomes small (note that $H \geq -z \geq 0$), the dispersion relation reduces to

$$\sigma = k\sqrt{gH}$$

and the solution to

$$p_1 = \rho g A \, cos(kx - \sigma t)$$

$$w_1 = A\,\frac{z+H}{H}\,sin(kx-\sigma t)$$

$$z_1 = A\,\frac{z+H}{H}\,cos(kx-\sigma t)$$

$$u_1 = \frac{A}{kH}\,cos(kx-\sigma t)$$

$$x_1 = \frac{-A}{kH}\,sin(kx-\sigma t)$$

The particle trajectories for these shallow water waves become very flat ellipses with a horizontal velocity and displacement that are nearly independent of depth (see Figure 35(b)) and become larger as H decreases.

The fact that the shallow water solution predicts a significant horizontal velocity at the sea floor is a consequence of the fact that we have ignored viscosity. However, the solution near the sea floor is just the oscillating plate that we considered earlier turned upside-down and shifted to a coordinate system fixed to the plate. In order to ignore the effect of the boundary it is necessary that the boundary layer thickness $\delta = \sqrt{\frac{2\nu}{\sigma}}$ be much smaller than the water depth. For 10 second period waves and molecular viscosity, $\delta \approx 1mm$. It would require an eddy viscosity of order 1 m^2/s to affect a depth of a meter. Such high eddy viscosities are conceivable in the surf zone, but not away from the shore.

The phase and group velocities for shallow water waves are

$$c_p = c_g = \sqrt{gH}$$

and are independent of frequency or wavelength. Thus these waves are non-dispersive because waves of all frequencies and wavelengths move at the same velocity. Non-dispersive waves are generated when the restoring force depends only on some inherent physical property of the material and not on the spatial gradients of displacement or velocity. Classic examples are light and radio waves in non-conducting media and sound and seismic body waves. Shallow water waves in a tank are a very useful analog for demonstrating the properties of these other waves.

An alternate derivation of the dispersion relation for shallow water wave follows from consideration of the $\hat{\mathbf{x}}$ component of the Navier-Stokes equation

$$\frac{\partial u_1}{\partial t} = -\frac{1}{\rho}\frac{\partial p_1}{\partial x} = -g\,\frac{\partial h}{\partial x}$$

where h is the vertical displacement and I have left off the viscous term for simplicity.

This equation is valid at all depths, but in particular is valid at the upper surface. We now vertically integrate the two-dimensional continuity equation to obtain

$$\int_{-H}^{0} \frac{\partial u_1}{\partial x}\, dz = H\, \frac{\partial u_1}{\partial x} = -\int_{-H}^{0} \frac{\partial w_1}{\partial z}\, dz = -\, w_1(0) = -\, \frac{\partial h}{\partial t}$$

where I have made use of the bottom boundary condition $w_1(-H) = 0$ and the fact that u_1 is independent of depth. The physical situation is simple: A miss-match of horizontal influx and outflux across the sides of a column of fluid will cause the top surface to rise or fall. We can now eliminate u_1 and get

$$\frac{\partial^2 h}{\partial t^2} - gH\, \frac{\partial^2 h}{\partial x^2} = 0$$

The dispersion relation $\sigma = k\sqrt{gH}$ falls out immediately when we substitute the two-dimensional plane wave $h = A\, cos(kx - \sigma t)$ into this equation. One can proceed to derive the pressure, velocities and displacements, but this approach has the disadvantage that one cannot connect it to the solution to deep water waves through an asymptotic analysis.

Shallow water waves in the open ocean are generated by storms (where they are known as storm surges or storm tides) and by earthquakes (where they are known as tsunamis). Resonant shallow water waves can be generated in bays and lakes by the same processes (where they are called sieches). Shallow water waves in the open ocean, which is about 4 km deep, travel at 200 m/s (700 km/hr). Their periods range from minutes to hours. Thus their wavelengths are tens to hundreds of kilometers. Their typical amplitudes in the open ocean are small (less than a meter), so they cannot be seen by eye until the conservation of energy and momentum result in their amplitudes rising to much larger values in shallower water.

However, by far the most common form of shallow water waves are simply deep water ocean waves that approach the coast. As we have already noted, a deep water wave must have $H > \frac{1}{k} \approx \frac{\lambda}{2}$. The equations for the phase velocities of deep and shallow water waves lead to the conclusion that the phase velocity of a deep water wave will always be faster than a shallow water wave of the same frequency (see Figure 36). Now any propagating wave must preserve its frequency as it moves into a region with different properties. You can easily understand this by asking yourself what would happen if the water surface crossed a sharp change in conditions and the frequency was not preserved. Then the surface just on either side of the change would oscillate up and down at different rates and a tear would develop, which is impossible. Thus, as the wave begins to feel the bottom, its phase velocity and wavelength must decrease. This explains the focusing of wave energy on a headland (see Figure 37) and predicts that coastlines will be straight unless there are regions of rock that are especially hard to erode. Second, since the total kinetic energy of the wave will remain constant, but be concentrated in a progressively thinner water layer, the fluid velocity and wave amplitude must increase. At some point, the rising fluid velocity and falling phase velocity will cause failure of the assumption that the non-linear terms are negligible. Non-linear waves tend to lean forward in the direction of propagation (i.e. their front is steeper than their rear). The shape of the crests and troughs also become asymmetric with the smooth top of the sine or cosine changing to a sharp peak with an apex angle of about 120^o (see Figure 38). Eventually the fluid velocity will attempt to exceed the phase velocity and a completely new kind of disturbance called a shock wave forms (i.e. the wave breaks).

Physically, we see the wave over-steepen so much in the front that the wave collapses on itself and forms a concentrated zone of turbulence that is called a breaker or a bore (see Figure 38). In front of the bore, the water is so shallow that the phase and group speed is less than the speed of the bore. The water in front of the bore knows nothing about the approaching bore because it is impossible to transmit information faster than the group velocity. Behind the bore, the water is deep enough that the phase and group speeds equals or exceed that of the bore. Waves coming in can overtake the bore. The result is that the disturbance created by the bore remains concentrated at the bore.

There is a great deal of dissipation of the wave energy in the turbulence of the bore, but note that this dissipation and the original breaking of the wave are not due to viscous effects between the water and the bottom. The bottom plays a dominant role only after the breaker runs up the beach and the fluid starts to penetrate the sand.

Note also that when the wave breaks, the momentum carried by the wave (which involves no net transport of mass - the water in the wave sloshes back and forth but on the average goes nowhere) is converted into the momentum of the bore which does involve net mass transport. In fact non-linear waves (even at second order) turn out to involve some net mass transport, because the term $\mathbf{u}_1 \cdot \nabla \mathbf{u}_1$ squares the first order velocity and thus acts like a rectifier in an electronic circuit: Its output is not symmetric with respect to its input. This mass transport has two consequences. The transport component normal to the shore line drives water up the beach. This shoreward transport must be balanced by a return flow, which usually is distributed along the beach and occurs primarily in the time period between the waves. Under some circumstances (typically big surf and rising tide) the return flow can vary along the shore in a process which is unstable. Regions of higher outflow increase erosion and lower the local beach height. The regions of greater beach height (i.e. where it is not eroded) make the waves non-linear at a greater distance from the coast and thus increase the shoreward transport. Thus the outward flow through the surf zone becomes concentrated in the lows which increases the local erosion further concentrating the outward current. This process sometimes develops what are called beach cusps which vary rhythmically along the coast. Other times, very strong and narrow outflows are generated which are called rip currents and can be dangerous to swimmers who do not appreciate their fundamentally narrow nature and are swept out to sea. The current component along the beach when the wave energy does not arrive exactly normal to the shoreline is called longshore drift. On coasts where there is a consistent non-normal direction to the arrival of wave energy, such currents can transport a great deal of sand along the shore. The building of breakwaters at the entrance of rivers has the effect of preventing the longshore drift from filling in a dredged channel. However, because it cuts off the supply of sand to other parts of the coast, these breakwaters can result in major erosion elsewhere. It is not uncommon for engineers to build breakwaters normal to the shore every hundred meters or so in an effort to stabilize a shoreline that has had its natural equilibrium disturbed in this manner.

6.2.5. Kelvin waves and tides

The picture often presented of tides as water bulges associated with the time varying gravity field due to orbital interaction between the rotating earth and the Moon and Sun is appropriate only for an Earth completely covered by an ocean of uniform depth. It is completely inappropriate for the real Earth and its ocean basins. The tides are actually resonant shallow water waves excited by the tidal gravity field. However, because their period is comparable to a day, one cannot neglect the Coriolis term in the momentum equation. If we consider the simple case of a wave motion that is still two-dimensional with motion confined to the x-z plane and propagation in the $\hat{\mathbf{x}}$ direction, the equations for shallow water waves become

$$\frac{\partial u_1}{\partial t} = -g\frac{\partial h}{\partial x}$$

$$fu_1 = -\frac{1}{\rho}\frac{\partial p_1}{\partial y} = -g\frac{\partial h}{\partial y}$$

$$H\frac{\partial u_1}{\partial x} = -\frac{\partial h}{\partial t}$$

where f is the Coriolis parameter The first and third equations are identical to our previous equations. Thus h still obeys

$$\frac{\partial^2 h}{\partial t^2} - gH\frac{\partial^2 h}{\partial x^2} = 0$$

which is satisfied by $h = A\cos(kx - \sigma t)$ and the dispersion relation is still $\sigma = k\sqrt{gH}$. However, satisfying the second equation requires that h and hence A depend on y. Eliminating $\mathbf{u}_1$ using the second and third equations we find

$$\frac{\partial^2 h}{\partial x \partial y} = \frac{f}{gH}\frac{\partial h}{\partial t}$$

Substituting the plane wave and using the dispersion relations gives

$$\frac{\partial A}{\partial y} + \frac{f}{\sqrt{gH}}A = 0$$

Thus

$$A = A_0 \, e^{-(\frac{f}{c_p})y}$$

The solution decays in a direction to the left hand of the phase velocity and grows in the opposite direction (see Figure 39). It thus cannot exist in an ocean of infinite horizontal extent. However, if the ocean has a coast at which the wave amplitude is maximum, it can decay into the basin. At a latitude of 45^o the horizontal scale for this decay is about 2000 km. This wave is known as a Kelvin wave and sometimes as an edge wave for obvious reasons. Kelvin waves will circulate in a counter-clockwise sense inside the circumference of a Northern Hemisphere basin with their maximum amplitude at the outer edge (see Figure 40(a)). If the time for this circulation corresponds to one day (as it does in the North Atlantic) then you expect one rotating crest and one trough (see Figure 40(c)). The actual amplitude at the coastline is much larger than in the open ocean due to the conservation of mass flux as the wave moves into the much shallower continental shelf water (see Figure 40(b)). If the time for propagation around the outer circumference is much longer than a day (as it is in the Pacific) one might expect multiple crests and troughs. In actuality, however, the ocean breaks up into multiple regions each of which has a single rotating crest and trough circulating about a nodal point called an amphidrome (see Figure 40(c)).

6.3. Capillary waves

The effect of surface tension at the free water surface is to add a restoring force that is proportional to the curvature of the surface. Thus, for the two-dimensional case, the total restoring force due to gravity and surface tension is well-approximated by

$$@z = 0: \qquad p_1 = \rho g A h - T \frac{\partial^2 h}{\partial x^2}$$

where T is an empirical constant. Following the steps for deep water waves, this leads to a kinematic boundary condition

$$@\ z = 0: \qquad \frac{\partial^2 p_1}{\partial t^2} + g \frac{\partial p_1}{\partial z} + \frac{T}{\rho} \frac{\partial^3 p_1}{\partial z \partial x^2} = 0$$

The pressure still satisfies Laplace's equation and we can easily show that the dispersion relation becomes

$$\sigma = \pm \sqrt{gk + \frac{T}{\rho} k^3}$$

We see that the surface tension term will always dominate the gravity term for sufficiently large k (small wavelength). Furthermore, if we ignore gravity, the phase and group velocities can easily be shown to be

$$c_p = \frac{2}{3} c_g = \sqrt{\frac{T}{\rho} k}$$

These so-called ‘‘capillary’’ waves differ in two important ways from deep water surface waves: their phase and group velocities decrease as the wavelength increases and the group velocity exceeds the phase velocity. Thus you expect to find short waves at the leading edge of a wave group and the group will continuously overtake individual crests. The ripples that are generated when a droplet falls into still water are capillary waves and it is the fact that the short waves propagate faster than long waves that gives a spreading circular ripple its distinctive character. You should actually be able to see the long waves behind the initial ripple if you look closely.

In water, there is a transition between the decreasing phase speed of the capillary waves and the increasing phase speed of gravity waves which results in a minimum phase speed of 23 cm/s at a wavelength of 1.7 cm (see Figure 41)). This observation can be used to determine T.

Shallow water capillary waves are theoretically possible, but the necessary water depth becomes comparable to the viscous boundary thickness due to the higher frequency of capillary waves. You may have noticed what appear to be waves in thin sheets of water flowing down inclines. These are actually shear flow instabilities in the Poiseuille flow. Their small scale means that they are heavily influenced by surface tension and are not quite the same as the Kelvin-Helmholtz shear flow instabilities seen at larger scales (see Tritton, section 16.7).

Capillary waves play a very important role in the growth of gravity waves when wind blows over water. The ‘‘cat’s paws’’ seen when a puff of wind effects the surface of calm water are capillary waves. The ripples initially grow due to the Venturi and Kelvin-Helmholtz instabilities (mentioned in Chapter 5) until they become non-linear. As discussed earlier, non-linearity results in the velocity field being multiplied by itself. This squares and hence ‘‘rectifies’’ the velocity field which results in transferring energy into longer wavelengths. This process is a cascade. Each progressively larger scale grows both by the non-linear rectification process and by the Kelvin-Helmholtz instability. The cascade of energy from smaller to larger scales is most rapid when each scale actually breaks (forming whitecaps) as any sailor can tell you from experience. As noted earlier, the energy cascade stops when the wavelengths become long enough that non-linear effects are no longer significant.