Equivalence of Equations to Conservation Laws

Consider an isolated system:

total Mass $M = \int d^3r\, \rho$

$$\frac{dM}{dt} = \int d^3r\, \frac{\partial \rho}{\partial t} = -\int d^3r\, \nabla \cdot \rho \bar{V} = -\int d\bar{S} \cdot \rho \bar{V}$$

where the surface integral is over some surface completely surrounding the system. Since the system is assumed isolated, the surface can be taken far enough out so $\rho \bar{V} = 0$ on the surface $\therefore \frac{dM}{dt} = 0$ i.e. Mass is conserved

Thus the equation for $\frac{\partial \rho}{\partial t}$ is equivalent to conservation of mass.

In general, a conservation law will have the form

$$\frac{\partial}{\partial t}\left(\text{density of conserved quantity}\right) + \nabla \cdot \left(\begin{array}{c}\text{flux density}\\ \text{of conserved}\\ \text{Quantity}\end{array}\right) = 0$$

This insures that for an isolated system

$$\frac{d}{dt}\left(\text{total amount of conserved Quantity}\right) =$$

$$= \int d^3r\, \frac{\partial}{\partial t}\left(\text{density}\right) = \int -d\bar{S} \cdot \left(\text{flux}\right) = 0$$

Conservation of Momentum

Use Maxwell's equations to transform electromagnetic force density:

$$\rho_c \bar{E} + \bar{J}\times\bar{B} = \epsilon_0 \vec{E}(\bar{\nabla}\cdot\bar{E}) - \vec{B}\times\left(\frac{\bar{\nabla}\times\bar{B}}{\mu_0} - \epsilon_0 \frac{\partial}{\partial t}\vec{E}\right) =$$

$$= \left\{ \epsilon_0 \nabla\cdot(\bar{E}\bar{E}) - \epsilon_0(\bar{E}\cdot\bar{\nabla})\bar{E} - \nabla\frac{B^2}{2\mu_0} + \frac{(B\cdot\nabla)\bar{B}}{\mu_0} + \epsilon_0\frac{\partial}{\partial t}(\bar{B}\times\bar{E}) + \epsilon_0 \bar{E}\times\frac{\partial}{\partial t}\bar{B} \right\}$$

$$= \left\{ \frac{\nabla\cdot\bar{B}\bar{B}}{\mu_0} - \frac{\nabla B^2}{2\mu_0} + \epsilon_0\bar{\nabla}\cdot(\bar{E}\bar{E}) + \epsilon_0\frac{\partial}{\partial t}(\bar{B}\times\bar{E}) - \epsilon_0(\bar{E}\cdot\bar{\nabla})\bar{E} - \epsilon_0\bar{E}\times(\bar{\nabla}\times\bar{E}) \right\}$$

but $-(\bar{E}\cdot\bar{\nabla})\bar{E} - \bar{E}\times(\bar{\nabla}\times\bar{E}) = -\nabla\frac{E^2}{2} + (\bar{E}\cdot\bar{\nabla})\bar{E} - (\bar{E}\cdot\bar{\nabla})\bar{E}$

$$\therefore \boxed{(\rho_c\vec{E} + \bar{J}\times\bar{B}) = -\epsilon_0\frac{\partial}{\partial t}(\bar{E}\times\bar{B}) + \nabla\cdot\overleftarrow{\Gamma}_M}$$

where $\overleftarrow{\Gamma}_M \Rightarrow$ is the Maxwell stress tensor

$$\boxed{\overleftrightarrow{\Gamma}_M = \epsilon_0\bar{E}\bar{E} - \frac{\epsilon_0 E^2}{2}\overleftarrow{1} + \frac{\bar{B}\bar{B}}{\mu_0} - \frac{B^2}{2\mu_0}\overleftarrow{1}}$$

note that $\nabla\cdot B^2\bar{1} = \frac{\partial}{\partial x_i} B^2\delta_{ij} = \frac{\partial}{\partial x_j} B^2 = \nabla B^2$ etc

For Gravitational force, use Newton's $\frac{1}{r^2}$ law in form $\nabla\cdot\vec{g} = -G\rho$ G = gravitational constant

$\nabla\times\bar{g} = 0$

$$\therefore \rho\bar{g} = -\frac{1}{G}(\bar{g}\nabla\cdot\bar{g})$$

$$\bar{g}\,\nabla\cdot\bar{g} = \bar{\nabla}\cdot(\bar{g}\,\bar{g}) - (\bar{g}\cdot\nabla)\bar{g} = \nabla\cdot(\bar{g}\bar{g}) - \nabla\frac{g^2}{2} + \vec{g}\times(\nabla\times\vec{g})$$

$$\therefore\ \boxed{\rho\bar{g} = \nabla\cdot\overline{\overline{T}}_G \quad \text{where } \overline{\overline{T}}_G = -\frac{1}{G}\left(\bar{g}\,\bar{g} - \frac{1}{2}g^2\,\overline{\overline{I}}\right)}$$

gravitational stress tensor

$\therefore$ equation for $\frac{\partial}{\partial t}\rho\bar{V}$ can be written

$$\boxed{\frac{\partial}{\partial t}\left(\rho\bar{V} + \epsilon_0\,\bar{E}\times\bar{B}\right) + \nabla\cdot\left(\rho\bar{V}\bar{V} + \overline{\overline{P}} - \overline{\overline{T}}_M - \overline{\overline{T}}_G\right) = 0}$$

ie total momentum, mechanical + electromagnetic is conserved

Thus $\epsilon_0(\bar{E}\times\bar{B})$ is identified as the momentum density of the electromagnetic field. (The gravitational field has no momentum density in the Newtonian approximation.)

$\overline{\overline{T}}_M$ represents the momentum flux density of the electromagnetic field, and $\overline{\overline{T}}_G$ the momentum flux density of the gravitational field

Conservation of Energy

$$\bar{E}\cdot\bar{J} = \bar{E}\cdot\left(\frac{\bar{\nabla}\times B}{\mu_0} - \epsilon_0\frac{\partial}{\partial t}\bar{E}\right)$$

but $\bar{\nabla}\cdot\frac{(\bar{E}\times\bar{B})}{\mu_0} = -\frac{\bar{E}}{\mu_0}\cdot(\bar{\nabla}\times\bar{B}) + \frac{\bar{B}\cdot(\bar{\nabla}\times\bar{E})}{\mu_0}$

$$\therefore\ \bar{E}\cdot\bar{J} = -\frac{1}{\mu_0}\nabla\cdot(\bar{E}\times\bar{B}) - \frac{\bar{B}}{\mu_0}\cdot\frac{\partial}{\partial t}\bar{B} - \epsilon_0\bar{E}\cdot\frac{\partial\bar{E}}{\partial t}$$

or $\boxed{-\bar{E}\cdot\bar{J} = \frac{\partial}{\partial t}\left(\frac{\epsilon_0 E^2 + B^2/\mu_0}{2}\right) + \nabla\cdot\left(\frac{\bar{E}\times\bar{B}}{\mu_0}\right)}$ Poynting Theorem

$$\frac{\bar{E}\times\bar{B}}{\mu_0} = \text{poynting flux} \quad (= \bar{E}\times\bar{H})$$

for Gravitational field $\bar{g} = -\nabla\Phi_G$; Φ_G = Gravitational potential

$$\rho\bar{V}\cdot\bar{g} = -\rho\bar{V}\cdot\nabla\Phi_G = -\nabla\cdot\rho\bar{V}\Phi_G + \Phi_G\,\nabla\cdot\rho\bar{V} =$$

$$= -\bar{\nabla}\cdot\rho\,\Phi_G - \Phi_G\frac{\partial\rho}{\partial t} = -\nabla\cdot\rho\bar{V}\Phi_G - \frac{\partial}{\partial t}\rho\Phi_G + \rho\frac{\partial\Phi_G}{\partial t}$$

$$\therefore\ \boxed{\text{if } \frac{\partial\Phi_G}{\partial t} = 0,\quad -\rho\bar{V}\cdot\bar{g} = \frac{\partial}{\partial t}\rho\Phi_G + \nabla\cdot(\rho\Phi_G\bar{V})}$$

Work done by gravitational field can be expressed as local conservation of energy only if the gravitational potential doesnot vary in time.

This is because the Newtonian approximation does not describe the energy flux of the gravitational field in time-varying situations (for that, need general relativity)

Thus, assuming $\rho \frac{\partial \Phi_G}{\partial t}$ is negligible

$$\boxed{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + U_T + \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} + \rho \Phi_G\right) + \nabla \cdot \left[\left(\frac{1}{2}\rho v^2 + U_T + \rho \Phi_G\right)\bar{v} + \bar{\bar{P}} \cdot \bar{v} + \bar{q} + \frac{\bar{E} \times \bar{B}}{\mu_0}\right] = 0}$$

Conservation of energy: mechanical plus electromagnetic (with some restrictions) also gravitation.

Thus $\frac{\epsilon E^2}{2} + \frac{B^2}{2\mu_0}$ is identified as the energy density of the electromagnetic field and the Poynting flux $\frac{E \times B}{\mu_0}$ as the energy flux density of the EM field.

So far we have developed equations for time derivatives at a fixed point in space. It is often of interest to consider the time derivatives following the bulk flow of the plasma, i.e. the time variation seen if one moves with the velocity $\bar{V}$,

The time derivative following the flow is the convective (or hydromagnetic) derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \bar{V}\cdot\bar{\nabla}$$

continuity equation $\frac{\partial}{\partial t}\rho + \bar{\nabla}\cdot\rho\bar{V} = \frac{\partial}{\partial t} + (\bar{V}\cdot\nabla)\rho + \rho\bar{\nabla}\cdot\bar{V} = 0$

$$\therefore \boxed{\frac{\partial}{\partial t}\rho + \rho\bar{\nabla}\cdot\bar{V} = 0}$$

momentum equation $\frac{\partial}{\partial t}\rho\bar{V} + \nabla\cdot(\rho\bar{V}\bar{V}) =$

$$\rho\frac{\partial\bar{V}}{\partial t} + \rho\bar{V}\cdot\nabla\bar{V} + \bar{V}\left\{\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\bar{V}\right\}$$

but $\{\,\} = 0$ by continuity eqn

$$\therefore \boxed{\rho\frac{d\bar{V}}{dt} + \nabla\cdot\overleftarrow{\underline{P}} = \rho_c\bar{E} + \bar{J}\times\bar{B} + \rho\vec{g}}$$

Energy equation: Interested in obtaining $\frac{d}{dt}U_T$

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho V^2 + U_T\right) + \bar{\nabla}\cdot\left[\left(\tfrac{1}{2}\rho V^2 + U_T\right)\bar{V}\right] + \frac{\partial}{\partial r_i}\left(P_{ij}V_j\right) + \nabla\cdot\bar{q} = \bar{E}\cdot\bar{J} + \rho\bar{V}\cdot\bar{g}$$

→ over

Writing $\nabla \cdot U_T \bar{V} = \bar{V} \cdot \nabla U_T + U \bar{\nabla} \cdot \bar{V}$ etc
and $\frac{\partial}{\partial r_i} P_{ij} V_j = V_j \frac{\partial}{\partial r_i} P_{ij} + P_{ij} \frac{\partial V_j}{\partial r_i}$

$$\frac{d}{dt}\left(\tfrac{1}{2}\rho V^2 + U_T\right) = \left(\tfrac{1}{2}\rho V^2 + U_T\right)\bar{\nabla}\cdot\bar{V} + \bar{V}\cdot(\nabla\cdot\underline{\vec{P}}) +$$

$$+ \underline{P}_{ij}\frac{\partial V_j}{\partial x_i} + \nabla\cdot\vec{q} = \bar{E}\cdot\bar{J} + \rho\bar{V}\cdot\bar{g}$$

introduce $\bar{E}^*, \bar{J}^*$ referred to the frame of reference moving with $\bar{V}$

$$E^* = \bar{E} + \bar{V}\times\bar{B} \qquad \vec{J}^* = \vec{J} - \rho_c\bar{V}$$

$$\vec{E}^*\cdot\vec{J}^* = \bar{E}\cdot\bar{J} - \bar{V}\rho_c E - \bar{V}\cdot\bar{J}\times\bar{B}$$

insert into eqtn. above and rearrange terms,
also recall $\frac{1}{2}\frac{d}{dt}V^2 = \bar{V}\cdot\frac{d\bar{V}}{dt}$

$$\tfrac{1}{2}V^2\left\{\frac{d\rho}{dt} + \rho\bar{\nabla}\cdot\bar{V}\right\} + \vec{V}\cdot\left\{\rho\frac{d\bar{V}}{dt} + \nabla\cdot\bar{P} - \rho_c\vec{E} - \bar{J}\times\vec{B} - \rho\vec{g}\right\}$$

$$+ \frac{d}{dt}U_T + U_T\bar{\nabla}\cdot\bar{V} + P_{ij}\frac{\partial V_j}{\partial x_i} = \vec{E}^*\cdot\vec{J}^* - \nabla\cdot\vec{q}$$

The first $\{\ \} = 0$ by continuity eqtn
The second $\{\ \} = 0$ by momentum eqtn
write pressure Tensor as

$$P_{ij} = \underline{P}\,\delta_{ij} + \mathcal{S}_{ij}$$

where $\mathcal{S}_{ij}$ is chosen to have zero trace i.e. $P_{kk} = 3P$

$\therefore\ \mathcal{S}_{ij} = P_{ij} - \left(\frac{1}{3} P_{kk}\right)\delta_{ij}$ and $\mathcal{S}_{ii} = 0$

Then $P_{ij} \frac{\partial V_j}{\partial x_i} = P\, \bar{\nabla}\cdot\bar{V} + \mathcal{S}_{ij} \frac{\partial V_j}{\partial r_i}$

For isotropic pressure $\mathcal{S}_{ij} = 0$

also $U_T = \frac{1}{2} P_{ii} = \frac{3}{2} P$ $\qquad \nabla\cdot\bar{V} = -\frac{1}{\rho}\frac{\partial \rho}{\partial t}$

$\therefore\ \frac{3}{2}\frac{d}{dt} P - \frac{5}{2}\frac{P}{\rho}\frac{d\rho}{dt} = \bar{E}^{*}\cdot J^{*} - \nabla\cdot\bar{q} - \mathcal{S}_{ij}\frac{\partial v_i}{\partial r_j}$

or

$$\boxed{\frac{d}{dt}\log\left(\frac{P}{\rho^{5/3}}\right) = \frac{2}{3P}\left(\vec{E}^{*}\cdot\vec{J}^{*} - \nabla\cdot\bar{q} - \mathcal{S}_{ij}\frac{\partial V_i}{\partial r_j}\right)}$$

Thus $\frac{P}{\rho^{5/3}}$ is changed only by (a) work done by electric field in plasma center-of-mass frame of reference (this corresponds to ohmic heating in ordinary fluids),

(b) heat flux

(c) anisotropic pressure interacting with velocity gradient (This corresponds to work done by viscous forces in ordinary fluids).