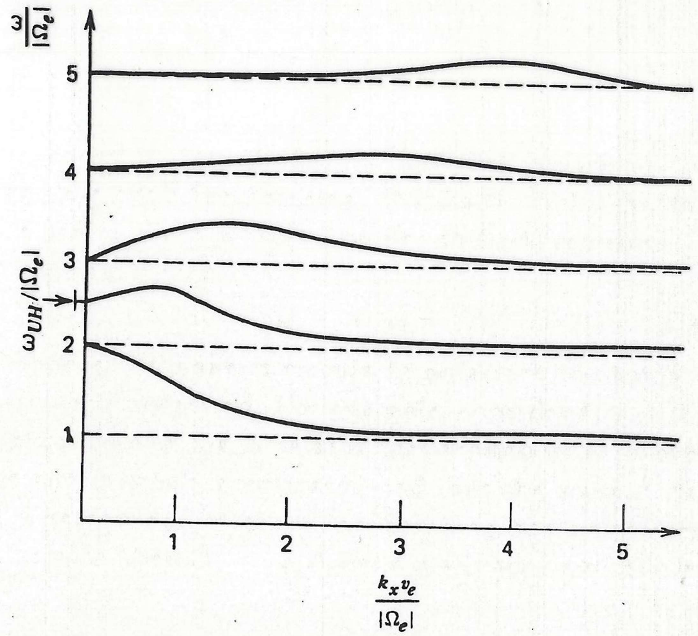

Fig. 6.33 Sketch of the dispersion curves for electron Bernstein modes.

When $l = 0$ in (6.191), we have our old friend Landau damping. When $l \neq 0$, we have *cyclotron damping*. Physically, cyclotron damping occurs when the particle sees a wave whose Doppler shifted frequency is the gyrofrequency or some harmonic thereof:

$$\omega - k_z v_z = l\Omega_s, \qquad l = \pm 1, \pm 2, \ldots . \tag{6.192}$$

Suppose the wave is circularly polarized, or has at least one component that is

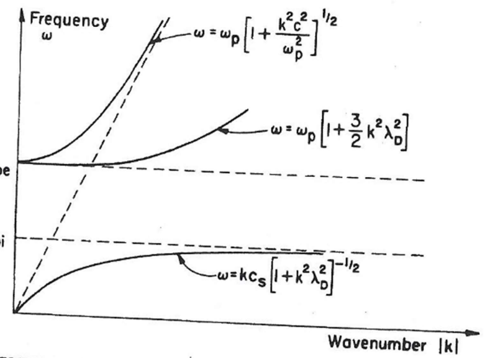

FIGURE 8.9.1
Dispersion relation for waves $\exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)]$ in a field-free plasma.

8.10 THE VLASOV THEORY OF SMALL-AMPLITUDE WAVES IN A UNIFORMLY MAGNETIZED PLASMA $[\mathbf{B}_0 = B_0\hat{\mathbf{z}}, \mathbf{E}_0 = 0, f_{\alpha 0} = f_{\alpha 0}(v_\perp{}^2, v_\parallel)]$

The orbits of particles in an equilibrium plasma are quite complicated if that equilibrium includes a magnetic field. This allows new wave modes associated with particle gyration, and results in an alteration of the waves indigenous to the field-free plasma. The fact that changing the equilibrium fields, or even simply changing the velocity distribution, can entirely change the pattern and behavior of the plasma oscillations is a constant barrier to the development of a "complete" description of a plasma. The motions of most other systems of coupled oscillators can be simply described in terms of a few "spring constants." For example, the bulk modulus of a neutral gas provides complete information about sound waves; by contrast, two otherwise identical plasmas, one maxwellian, $\exp(-v^2/\bar{v}^2)$, one Poisson, $\exp(-|v|/\bar{v})$, behave quite differently in regard to sound waves. This chameleon behavior of plasma makes it necessary to study each equilibrium separately. Fortunately, the same *methods* can be used in all these cases.

If the plasma is uniformly magnetized $(\mathbf{B}_0 = B_0\hat{\mathbf{z}})$, the Vlasov equation, linearized about f_0, becomes (writing $f = f_0 + f_1$, $\mathbf{B} = B_0\hat{\mathbf{z}} + \mathbf{B}_1$, $\mathbf{E} = \mathbf{E}_1$)

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \frac{q_\alpha}{m_\alpha}\frac{\mathbf{v}\times\mathbf{B}_0}{c}\cdot\nabla_\mathbf{v}\right)f_{\alpha 1} = -\frac{q_\alpha}{m_\alpha}\left(\mathbf{E}_1 + \frac{\mathbf{v}\times\mathbf{B}_1}{c}\right)\cdot\nabla_\mathbf{v} f_{\alpha 0} \qquad (8.10.1)$$

and the equilibrium distribution satisfies

$$\left(\mathbf{v}\cdot\nabla + \frac{q_\alpha}{m_\alpha}\frac{\mathbf{v}\times\mathbf{B}_0}{c}\cdot\nabla_\mathbf{v}\right)f_{\alpha 0} = 0$$

$$\rho_q = \sum_\alpha \bar{n}_\alpha q_\alpha \int f_{\alpha 0}\,d\mathbf{v} = 0 \qquad \text{no net charge in the plasma}$$

$$\mathbf{J} = \sum_\alpha \bar{n}_\alpha q_\alpha \int \mathbf{v} f_{\alpha 0}\,d\mathbf{v} = 0 \qquad \text{no net current in the plasma} \qquad (8.10.2)$$

For the case of a spatially uniform plasma, the most general solution of (8.10.2) which is isotropic in the plane perpendicular to $\mathbf{B}_0$ has the form

$$f_{\alpha 0} = f_{\alpha 0}(v_\perp{}^2, v_z) \qquad (8.10.3)$$

This section investigates waves that propagate in a plasma with an equilibrium distribution of this form. Since the motion of charged particles is so different along and across a magnetic field, it would be unrealistic to restrict $f_{\alpha 0}$ to be isotropic $[f_{\alpha 0} = f_{\alpha 0}(v^2)]$, as was done in the study of waves in field-free plasmas. For example, a realistic distribution of the form (8.10.3) might be

$$f_0 = \frac{m}{2\pi\kappa T_\perp}\left(\frac{m}{2\pi\kappa T_\parallel}\right)^{1/2}\exp\left[-\frac{m}{2\kappa}\left(\frac{v_\perp{}^2}{T_\perp} + \frac{v_\parallel{}^2}{T_\parallel}\right)\right]$$

where $v_\parallel = v_z$, $v_\perp = \sqrt{v_x{}^2 + v_y{}^2}$.

The Vlasov equation is solved by integrating along the orbits of the particles in the *unperturbed* fields, as described in Sec. 8.8. For the case considered here $(\mathbf{E}_0 = 0, \mathbf{B}_0 = B_0\hat{\mathbf{z}})$, these orbits are best expressed in cylindrical coordinates in velocity space; that is, $v_x = v_\perp\cos\phi$, $v_y = v_\perp\sin\phi$, $v_z = v_\parallel$. In terms of these variables the particle orbits $\mathbf{x}'(\tau)$ are

$$v_x' = v_\perp\cos(\phi - \omega_c\tau) \qquad x' = x - \frac{v_\perp}{\omega_c}\sin(\phi - \omega_c\tau) + \frac{v_\perp}{\omega_c}\sin\phi$$

$$v_y' = v_\perp\sin(\phi - \omega_c\tau) \qquad y' = y + \frac{v_\perp}{\omega_c}\cos(\phi - \omega_c\tau) - \frac{v_\perp}{\omega_c}\cos\phi \qquad (8.10.4)$$

$$v_z' = v_\parallel \qquad z' = v_\parallel\tau + z$$

These orbits are derived in Appendix I; here the constants of integration are chosen so that at $\tau\to 0$, $\mathbf{v}'\to\mathbf{v}$, $\mathbf{x}'\to\mathbf{x}$, where $\mathbf{v}$, $\mathbf{x}$ are fixed points in phase space.

Note that

$$f_0[\mathbf{x}'(\tau), \mathbf{v}'(\tau)]$$

is constant along the orbit of a particle in the unperturbed fields, because f_0 is constructed out of constants of the motion, as explained in Sec. 7.7.

The perturbed distribution, from (8.8.6), is

$$\bar{f}_{\alpha\mathbf{k}} = -\frac{q_\alpha}{m_\alpha}\int_{-\infty}^{0}\left(\bar{\mathbf{E}}_1 + \frac{\mathbf{v}' \times \bar{\mathbf{B}}_1}{c}\right)\cdot \nabla_{\mathbf{v}'} f_{\alpha 0}(\mathbf{v}')\exp[i(\mathbf{k}\cdot\mathbf{X} - \omega\tau)]\, d\tau$$

$$\operatorname{Im}(\omega) > 0 \qquad (8.10.5)$$

where $\mathbf{X} = \mathbf{x}' - x$, $\tau = t' - t$. The term $(\mathbf{v} \times \bar{\mathbf{B}}_1)\cdot\nabla_{\mathbf{v}} f_0$ vanishes if $f_0 = f_0(v^2)$. But as mentioned above, this is not a realistic assumption for plasma in a **B** field, since motions across and along the magnetic field are quite different. For a distribution of the form $f_0 = f_0(v_\perp{}^2, v_z)$, the field $\bar{\mathbf{B}}_1$ must be evaluated in terms of $\bar{\mathbf{E}}_1$ by Maxwell's equation

$$i\mathbf{k} \times \bar{\mathbf{E}}_1 = i\frac{\omega}{c}\bar{\mathbf{B}}_1$$

The operation $\nabla_{\mathbf{v}} f_0$ can be written

$$\nabla_{\mathbf{v}} f_0 = 2(\mathbf{v} - v_z\hat{\mathbf{z}})\frac{\partial f_0}{\partial v_\perp{}^2} + 2v_z\frac{\partial f_0}{\partial v_z{}^2}\hat{\mathbf{z}}$$

Because $v_\perp{}^2$ and v_z are constants of the motion, the terms $\partial f_0/\partial v_\perp{}^2$ and $\partial f_0/\partial v_z$ can be removed from the integral. All remaining integrals have the form

$$\int_{-\infty}^{0}(v_x', v_y', 1)\exp[i(\mathbf{k}\cdot\mathbf{X} - \omega\tau)]\, d\tau$$

where $\mathbf{X}$ $(=\mathbf{x}' - \mathbf{x})$, v_x', and v_y', are given by (8.10.4). Integrals of this form can be done with the aid of the following identity:

$$\exp\left[i\frac{k_\perp v_\perp}{\omega_c}\sin(\phi - \omega_c\tau)\right] = \sum_{n=-\infty}^{+\infty} J_n\left(\frac{k_\perp v_\perp}{\omega_c}\right)\exp[in(\phi - \omega_c\tau)] \qquad (8.10.6)$$

where J_n is the ordinary Bessel's function of the first kind.

Taking the x axis along

$$\mathbf{k}_\perp\left(\equiv \mathbf{k} - \frac{\mathbf{k}\cdot\mathbf{B}}{|\mathbf{B}|}\hat{\mathbf{z}}\right)$$

without any loss of generality,

$$\mathbf{k} = k_\perp\hat{\mathbf{x}} + k_\parallel\hat{\mathbf{z}} \qquad (8.10.7)$$

the result for $\bar{f}_{\alpha\mathbf{k}}$ is

$$\bar{f}_{\alpha\mathbf{k}} = \frac{q_\alpha}{m_\alpha}\sum_{n,l}\left[\frac{2Zv_\parallel J_l(k_\perp v_\perp/\omega_{c\alpha}) + Xv_\perp(J_{l+1} + J_{l-1}) - iYv_\perp(J_{l+1} - J_{l-1})}{i(\omega - l\omega_c - k_\parallel v_\parallel)}\right]$$

$$\cdot J_n\left(\frac{k_\perp v_\perp}{\omega_{c\alpha}}\right)\exp[i(n - l)\phi] \qquad (8.10.8)$$

where

$$X = E_x\frac{\partial f_{\alpha 0}}{\partial v_\perp{}^2} + \frac{v_\parallel}{\omega}(k_\parallel E_x - k_\perp E_z)\left(\frac{\partial f_{\alpha 0}}{\partial v_\parallel{}^2} - \frac{\partial f_{\alpha 0}}{\partial v_\perp{}^2}\right)$$

$$Y = E_y\frac{\partial f_{\alpha 0}}{\partial v_\perp{}^2} + \frac{v_\parallel}{\omega}k_\parallel E_y\left(\frac{\partial f_{\alpha 0}}{\partial v_\parallel{}^2} - \frac{\partial f_{\alpha 0}}{\partial v_\perp{}^2}\right)$$

$$Z = E_z\frac{\partial f_{\alpha 0}}{\partial v_\parallel{}^2}$$

The perturbed currents $\mathbf{J}_{\mathbf{k}}$ can be calculated from $\bar{f}_{\alpha\mathbf{k}}$, and then substituted into Maxwell's equations. The result is

$$-\mathbf{k} \times \mathbf{k} \times \bar{\mathbf{E}}_1 = \frac{\omega^2}{c^2}\bar{\mathbf{E}}_1 + \frac{i\omega}{c^2}4\pi\sum_\alpha \bar{n}_\alpha q_\alpha \int_L \mathbf{v}\bar{f}_{\alpha\mathbf{k}}\, d\mathbf{v} \qquad (8.10.9)$$

where L is the Landau contour (Figs. 8.4.2 to 8.4.4). The dispersion equation takes the form

$$\begin{vmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{vmatrix} = 0 \qquad (8.10.10)$$

where the elements of the determinant (8.10.10) are

$$D_{xx} = 1 - \frac{k_\parallel{}^2 c^2}{\omega^2} - \frac{2\pi}{\omega}\sum_\alpha\left(\frac{\omega_p{}^2}{\omega_c}\right)_\alpha\sum_n\left[\frac{n^2\omega_{c\alpha}{}^3}{k_\perp{}^2}J_n{}^2\chi_\alpha\right]$$

$$D_{xy} = -\frac{2\pi i}{\omega}\sum_\alpha\sum_n\left(\frac{\omega_p{}^2}{\omega_c}\right)_\alpha\left\{\frac{n\omega_c{}^2 v_\perp}{k_\perp}J_n\frac{dJ_n}{d(k_\perp v_\perp/\omega_{c\alpha})}\chi_\alpha\right\}$$

$$D_{xz} = \frac{k_\parallel k_\perp c^2}{\omega^2} - \frac{2\pi}{\omega}\sum_\alpha\sum_n\left(\frac{\omega_p{}^2}{\omega_c}\right)_\alpha\left(\frac{n\omega_{c\alpha}{}^2 v_\parallel J_n{}^2}{k_\perp}\Lambda_\alpha\right)$$

$$D_{yx} = -D_{xy}$$

$$D_{yy} = 1 - \frac{(k_\perp{}^2 + k_\parallel{}^2)c^2}{\omega^2} - \frac{2\pi}{\omega}\sum_\alpha\sum_n\left(\frac{\omega_p{}^2}{\omega_c}\right)_\alpha\left[\omega_{c\alpha}\left(\frac{dJ_n}{d(k_\perp v_\perp/\omega_{c\alpha})}\right)^2 v_\perp{}^2\chi_\alpha\right] \qquad (8.10.11)$$

$$D_{yz} = \frac{2\pi i}{\omega}\sum_\alpha\sum_n\left(\frac{\omega_p{}^2}{\omega_c}\right)_\alpha\left\{\omega_{c\alpha}v_\perp v_\parallel J_n\frac{dJ_n}{d(k_\perp v_\perp/\omega_{c\alpha})}\Lambda_\alpha\right\}$$

$$D_{zx} = \frac{k_\parallel k_\perp c^2}{\omega^2} - \frac{2\pi}{\omega}\sum_\alpha\sum_n\left(\frac{\omega_p{}^2}{\omega_c}\right)_\alpha\left[v_\parallel\frac{n\omega_{c\alpha}{}^2}{k_\perp}J_n{}^2\chi_\alpha\right]$$

$$D_{zy} = -\frac{2\pi i}{\omega}\sum_\alpha\sum_n\left(\frac{\omega_p{}^2}{\omega_c}\right)_\alpha\left\{v_\parallel v_\perp\omega_{c\alpha}J_n\frac{dJ_n}{d(k_\perp v_\perp/\omega_{c\alpha})}\chi_\alpha\right\}$$

$$D_{zz} = 1 - \frac{k_\perp{}^2c^2}{\omega^2} - \frac{2\pi}{\omega}\sum_\alpha\sum_n\left(\frac{\omega_p{}^2}{\omega_c}\right)_\alpha\left[v_\parallel{}^2\omega_{c\alpha}J_n{}^2\Lambda_\alpha\right]$$

In (8.10.11) the integral operator is defined by

$$[F(v)] \equiv \int_{-\infty}^{\infty} dv_{\parallel} \int_{0}^{\infty} \frac{2v_{\perp} F(v_{\perp}, v_{\parallel})}{k_{\parallel} v_{\parallel} + n\omega_{c\alpha} - \omega} dv_{\perp}$$

and two recurrent combinations of velocity derivatives are replaced by

$$\chi_{\alpha} \equiv \frac{\partial f_{\alpha 0}}{\partial v_{\perp}^{2}} \left(1 - \frac{k_{\parallel} v_{\parallel}}{\omega}\right) + \frac{k_{\parallel} v_{\parallel}}{\omega} \frac{\partial f_{\alpha 0}}{\partial v_{\parallel}^{2}}$$

$$\Lambda_{\alpha} \equiv \frac{\partial f_{\alpha 0}}{\partial v_{\parallel}^{2}} - \frac{n\omega_{c\alpha}}{\omega} \left(\frac{\partial}{\partial v_{\parallel}^{2}} - \frac{\partial}{\partial v_{\perp}^{2}}\right) f_{\alpha 0}$$

The argument of all Bessel functions above is $k_{\perp} v_{\perp}/\omega_{c\alpha}$, and the integrals $\int_{-\infty}^{\infty} dv$ are to be taken along the Landau contour shown in Fig. 8.4.1. Equations (8.10.10) and (8.10.11), which determine the wave motions of a magnetized plasma, are a depressing contrast to the relatively simple equations (8.9.6) and (8.9.7) which describe waves in a field-free plasma. The new features of the magnetized plasma are:

1. There are no longer three decoupled modes as in the field-free case. This is because the steady magnetic field **B** couples the motion of plasma particles from one direction to the others. That is, the plasma is anisotropic.
2. The separation into electrostatic and electromagnetic modes is no longer apparent.
3. There is now structure at $\omega \approx n\omega_c$. The gyration frequency of particles in a magnetic field $\omega_c (= eB/mc)$ is a natural frequency in this system, along with the plasma frequency ω_p.
4. The resonance at $\omega = \mathbf{k} \cdot \mathbf{v}$ found in field-free systems, which contributed damping of plasma waves, occurs at $\omega - n\omega_{c\alpha} = k_{\parallel} v_{\parallel}$; only particles moving along the magnetic field contribute to damping, and they damp only waves that have a component of propagation parallel to B_0. This is because in a uniform magnetic field there is no net motion of particles *across* the field.
5. In the field-free plasma the parameter which determines the oscillatory behavior is the ratio of the wave speed and the thermal speed,

$$\frac{\omega}{k}\sqrt{\frac{m_{\alpha}}{\kappa T_{\alpha}}}$$

In the magnetized plasma the parameters

$$\frac{\omega}{\omega_{c\alpha}} \qquad \frac{k}{\omega_{c\alpha}}\sqrt{\frac{\kappa T_{\alpha}}{m_{\alpha}}} \qquad \frac{\omega_p}{\omega_c} \qquad \frac{\omega}{k_{\parallel}}\sqrt{\frac{m_{\alpha}}{\kappa T_{\alpha}}} \qquad \frac{kc}{\omega_p}$$

all play a role in defining a parameter range.

To make some sense out of the general results (8.10.10) and (8.10.11), it is useful to single out some particular range for the parameters, looking always for ways in which the plasma resembles the field-free system and for ways in which the Vlasov results resemble the fluid approximation, as well as looking for distinctive new features.

8.11 THE VLASOV THEORY OF WAVES IN COLD MAGNETIZED PLASMA

The dispersion relation for waves in a cold magnetized plasma can be obtained from the dispersion relation derived using the linearized Vlasov equations by taking the limit $T_{\alpha} \to 0$ in (8.10.10). In this limit, (8.10.10) becomes identical with (4.9.6), the cold-magnetized-plasma dispersion determinant derived using the macroscopic-fluid-theory plasma equations. This shows that fluid theory provides an adequate description of waves in a sufficiently cold plasma, i.e., one where there is very little spread in the velocity distribution of the particles, so that all particles move with a speed nearly equal to that of a fluid element. In the cold-plasma limit there is no need for an equation of state; the pressure is simply neglected.

Since the solution of the linearized Vlasov equations reduced to the solution obtained using the macroscopic fluid theory, a comparison of the Vlasov theory results for $T_{\alpha} \neq 0$ with the results for $T_{\alpha} = 0$ identifies those plasma phenomena that are outside the scope of macroscopic fluid theory. For example, Landau damping, ion waves, and waves at harmonics of the cyclotron frequency, all depend on plasma temperature.

Problem 8.11.1 Show that as $T_{\alpha} \to 0$, (8.10.10), and (8.10.11) reduce to (4.9.6) and (4.9.9).
Hint: It is necessary to use the approximation

$$J_n \xrightarrow[T_{\alpha} \to 0]{} \frac{1}{n!}\left(\frac{k_{\perp} v_{\perp}}{2\omega_{c\alpha}}\right)^{n} \qquad ////$$

8.12 WAVES THAT PROPAGATE PERPENDICULAR TO THE EQUILIBRIUM MAGNETIC FIELD IN A HOT MAGNETIZED PLASMA ($\mathbf{E}_0 = 0$, $\mathbf{B}_0 = \hat{z}B_0$)—ELECTROMAGNETIC WAVES AND THE BERNSTEIN MODES

Taking $k_z = 0$ in (8.10.10) singles out the waves that propagate exactly perpendicular to the equilibrium magnetic field in a hot magnetized plasma. These

Bessel Function of the First Kind

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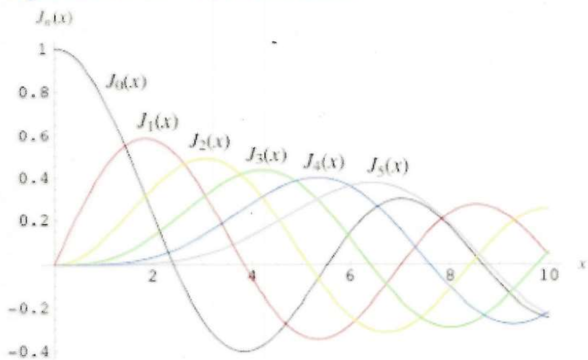

The Bessel functions of the first kind $J_n(x)$ are defined as the solutions to the Bessel differential equation

$$x^2 \frac{d^2 y}{d x^2} + x \frac{d y}{d x} + (x^2 - n^2) y = 0 \qquad (1)$$

which are nonsingular at the origin. They are sometimes also called cylinder functions or cylindrical harmonics. The above plot shows $J_n(x)$ for $n = 0$, 1, 2, ..., 5. The notation $J_{z,n}$ was first used by Hansen (1843) and subsequently by Schlömilch (1857) to denote what is now written $J_n(2z)$ (Watson 1966, p. 14). However, Hansen's definition of the function itself in terms of the generating function

$$e^{z(t-1/t)/2} = \sum_{n=-\infty}^{\infty} t^n J_n(z). \qquad (2)$$

is the same as the modern one (Watson 1966, p. 14). Bessel used the notation I_k^h to denote what is now called the Bessel function of the first kind (Cajori 1993, vol. 2, p. 279).

The Bessel function $J_n(z)$ can also be defined by the contour integral

$$J_n(z) = \frac{1}{2\pi i} \oint e^{(z/2)(t-1/t)} t^{-n-1} dt,$$