Alternate derivation of Harris solution

1. Overview

Harris assumed a 1D time-independent problem and then constructed a distribution function from invariants, automatically satisfying the Vlasov equation. By linearity, a plasma consisting of linear combinations of such solutions also satisfies the Vlasov equation. Requiring the total charge and current densities to be consistent with the electromagnetic field of the Vlasov equation imposes further conditions on the combination of distribution functions. The most accessible solution that emerges from this analysis is now called the Harris solution, in which the single component magnetic field is given by $B_x(z) = B\tanh(z/z_0)$.

Here the approach is different in that the starting point is an assumed distribution function containing undetermined functions. Requiring the distribution function to satisfy the Vlasov equation imposes constraints on the undetermined functions. From here, distribution functions can be combined so as to be consistent with Coulomb's law and Ampere's law.

2. Vlasov equation

Assume the model is time-independent and one-dimensional so that the electric and magnetic fields are $\mathbf{E}= (0, E, 0)$ and $\mathbf{B}= (B_x(z), 0, B_z)$. We seek solutions to Vlasov's equation having the form

$$f_\alpha(z, \mathbf{v}) = \frac{n_\alpha(z)}{(\sqrt{\pi}v_\alpha)^3}\exp\left(-\left(\frac{\mathbf{v}-\mathbf{V}_\alpha(z)}{v_\alpha}\right)^2\right),$$

where $v_\alpha = \sqrt{2k_BT_\alpha/m_\alpha}$ is a thermal speed. With these assumptions, the Vlasov equation is

$$\left(v_z\partial_z + \frac{q_\alpha}{m_\alpha c}\left(v_yB_z\partial_{v_x} + (v_zB_x(z) - v_xB_z)\partial_{v_y} - v_yB_x(z)\partial_{v_z}\right)\right)f_\alpha = 0.$$

Substituting and expanding yields the equation,

$$\frac{k_BT_\alpha}{m_\alpha}v_z\frac{n'_\alpha}{n_\alpha} = \frac{q_\alpha B_x}{m_\alpha c}v_yV_{z\alpha} - v_xv_zV'_{x\alpha} + V_{x\alpha}\left(v_zV'_{x\alpha} - \frac{q_\alpha B_z}{m_\alpha c}v_y\right) - v_yv_zV'_{y\alpha} +$$
$$V_{y\alpha}\left(\frac{q_\alpha B_z}{m_\alpha c}v_x - \frac{q_\alpha B_x}{m_\alpha c}v_z + v_zV'_{y\alpha}\right) + v_z(V_{z\alpha} - v_z)V'_{z\alpha}.$$

One obtains constraints by setting the coefficients of linearly independent combinations of $\mathbf{v}$ variables to zero. The v_xv_z, v_yv_z, and v_z^2 coefficients require $\mathbf{V}_\alpha$ to be constant. The v_x coefficient implies either $V_{y\alpha} = 0$ or $B_z = 0$. In the former trivial case, the v_z and v_y coefficients require $n_\alpha(z) = n_\alpha$ and $\mathbf{B} \propto \mathbf{V}_\alpha$, respectively. The latter ($B_z = 0$) case, through the v_y and v_z coefficients, gives $V_{z\alpha} = 0$ and $n_\alpha(z) = n_\alpha\exp\left(V_{y\alpha}q_\alpha A(z)/(ck_BT_\alpha)\right)$, where we introduced the vector potential through $B_x(z) = -A'(z)$.

3. Field equations

Coulomb's law, $\nabla\cdot\mathbf{E}= 4\pi\rho$, combined with the assumed electric field, requires that the charge density vanishes. Ampere's law, together with the assumed magnetic field, requires $J_x = 0$ and $J_y = -A''(z)c/(4\pi)$. Computing these source terms from distribution functions, this means

$$0 = \sum_\alpha q_\alpha n_\alpha\exp\left(\frac{V_{y\alpha}q_\alpha A(z)}{ck_BT_\alpha}\right)$$
$$0 = \sum_\alpha q_\alpha V_{x\alpha}n_\alpha\exp\left(\frac{V_{y\alpha}q_\alpha A(z)}{ck_BT_\alpha}\right)$$
$$-\frac{c}{4\pi}A''(z) = \sum_\alpha q_\alpha V_{y\alpha}n_\alpha\exp\left(\frac{V_{y\alpha}q_\alpha A(z)}{ck_BT_\alpha}\right).$$

For a 2-species plasma, the first equation requires that $q_1n_1 = -q_2n_2$ and that $q_1V_{y1}/T_1 = q_2V_{y2}/T_2$; then the second equation requires $V_{x1} = V_{x2}$; and then the third equation can be re-written as $A''(z) = c_1\exp(c_2A(z))$, which gives the Harris solution, $B_x(z)$ a hyperbolic tangent function. There are no other 2-component solutions of the form postulated in section 2.