5. Rotating Flow

The general circulation of the ocean (and atmosphere) differs markedly from the flows we typically observe around us and from the solutions that we have discussed so far, because large scale geophysical flows are: (1) dominated by the effects of rotation and (2) are almost inviscid. In this chapter I will introduce a technique to help us decide what terms in the Navier-Stokes equations are important in a given situation, present additional basic concepts and finally apply all that we have learned to the problem of the surface circulation of the oceans.

5.1. Non-dimensionalization

In the last chapter, we saw how approximate analyses of partial differential equations can provide insight into the behavior of the solution. Scaling the equations with typical values of the variables also has other uses, most of which begin with the basic concept of non-dimensionalization.

But before I discuss non-dimensionalization, I want to first consider the effect of variable density on the Navier-Stokes equation. Let

$$p \to p_H + \hat{p} \qquad\qquad \rho \to \rho(z) + \hat{\rho}$$

where p_H and $\rho(z)$ satisfy the hydrostatic equation

$$\nabla p_H = -\rho(z) g \hat{\mathbf{z}}$$

If we subtract this equation from the Navier-Stokes equation

$$(\rho(z) + \hat{\rho}) \frac{D\mathbf{u}}{Dt} = -\nabla(p_H + \hat{p}) - (\rho(z) + \hat{\rho}) g \hat{\mathbf{z}} + \mu \nabla^2 \mathbf{u}$$

and divide by $\rho(z) + \hat{\rho}$ we obtain

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho + \hat{\rho}} \nabla \hat{p} - \frac{\hat{\rho}}{\rho + \hat{\rho}} g \hat{\mathbf{z}} + \nu \nabla^2 \mathbf{u}$$

Note that gravity enters only to the extent that the density deviates from the hydrostatic situation. In must cases, $\hat{\rho} \ll \rho(z)$ and $\dfrac{1}{\rho + \hat{\rho}} = \dfrac{1 - \hat{\rho}}{\rho}$. Since this term only multiplies terms that already contain the perturbations $\hat{p}$ and $\hat{\rho}$, we can, to first order in the perturbations, write the full Navier-Stokes equations in a rotating system

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + 2\Omega \times \mathbf{u} = -\frac{1}{\rho} \nabla \hat{p} - \frac{\hat{\rho}}{\rho} g \hat{\mathbf{z}} + \nu \nabla^2 \mathbf{u}$$

The reduced gravity $\dfrac{\hat{\rho}}{\rho} g$ is often called the buoyancy.

Now let τ and L be typical time and length scales for the phenomenon that we are interested in. (I shall keep things simple at this point by assuming that the length scales are the same in all three coordinate directions.) Also let U, Π and $\Delta\rho$ be typical variations of velocity and pressure and density perturbation. We shall make the following changes of variables:

$$t = \tau t'$$

$$(x, y, z) = L(x', y', z')$$

$$\mathbf{u} = U\mathbf{u}'$$

$$\hat{p} = \Pi p'$$

$$\sigma \equiv \frac{\hat{\rho}}{\rho} g = \frac{\Delta\rho}{\rho} g\sigma'$$

where σ is the buoyancy. Note that all the primed variables are non-dimensional. Substituting into the Navier-Stokes equation we obtain

$$\frac{U}{T}\frac{\partial \mathbf{u}'}{\partial t'} + \frac{U^2}{L}\mathbf{u}' \cdot \nabla'\mathbf{u}' + 2\Omega U\,\hat{k} \times \mathbf{u}' + \frac{\Pi}{\rho L}\nabla' p' = -\frac{\Delta\rho}{\rho} g\sigma'\hat{\mathbf{z}} + \frac{\nu U}{L^2}\nabla'^2\mathbf{u}'$$

where I have used $\mathbf{\Omega} = \Omega\hat{\mathbf{k}}$ in order to include the possibility that the direction of $\mathbf{\Omega}$ is not the same as the direction of gravity. Historically this equation has been divided by the coefficient $\frac{U^2}{L}$ of the advection term. This was motivated by a desire to compare each of the linear terms to the non-linear one. Doing this division we get

$$\left[\frac{L}{UT}\right]\frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla'\mathbf{u}' + \left[\frac{2\Omega L}{U}\right]\hat{k} \times \mathbf{u}' + \left[\frac{\Pi}{\rho U^2}\right]\nabla' p' = -\left[\frac{\Delta\rho g L}{\rho U^2}\right]\sigma'\hat{\mathbf{z}} + \left[\frac{\nu}{UL}\right]\nabla'^2\mathbf{u}'$$

Each of the terms in this equation is non-dimensional and most of the coefficients in square brackets have acquired names:

$$\frac{L}{UT} = Strouhal\ number = St$$

$$\frac{2\Omega L}{U} = \frac{1}{Rossby\ number} = \frac{1}{Ro}$$

$$\frac{\Delta\rho g L}{\rho U^2} = \frac{1}{(internal\ Froude\ number)^2} = \frac{1}{Fr^2}$$

$$\frac{\nu}{LU} = \frac{1}{Reynolds\ number} = \frac{1}{\text{Re}}$$

The number $\frac{\Pi}{\rho U^2}$ has no name because its value is commonly determined by the flow itself. Note that the pressure perturbation has the units of energy per unit volume. Twice the kinetic energy per unit volume, ρU^2 is often called the dynamic pressure. There are lots of other ''numbers'' honoring pioneers in the field of fluid mechanics. They are all non-dimensional ratios. One that you have probably already heard of is the Mach number, which is the ratio of the fluid velocity to the speed of sound.

5.2. Geostrophic balance

The values of the non-dimensional numbers have at least two uses. First of all, their relative sizes tell you which terms are important in the Navier-Stokes equation. For instance, suppose you want to study the large scale, long term circulation of the North Atlantic Ocean. Figure 18 is a map of the average circulation of the oceans. The basic feature of the North Atlantic is a clockwise gyre which is much more intense along the western boundary. This western boundary current in the North Atlantic is called the Gulf Stream (GS). Its equivalent in the North Pacific is called the Kuroshio by the Japanese and the Black Current by the Chinese. The Southern Hemisphere oceans have similar gyres which rotate counterclockwise, but still have western boundary intensification. The only ocean that differs substantially from this picture is the Antarctic Ocean in which the eastward flowing Antarctic Circumpolar Current (AACP) flows completely around Antarctica. The velocities range from about 10 cm/s to more than 1 m/s. The currents are not entirely steady, but when averaged over many months are reasonably so. In calculating the non-dimensional numbers we want to use estimated scales that are correct to about an order of magnitude. We shall therefore take $L \approx 5000\ km$ (the width of an ocean, although one might also want to ask what would happen if we use the width of the GS, $L \approx 100$ km), $\tau \approx 1$ year, and $U \approx 30$ cm/s. We get $St \approx 0.5$, $\frac{1}{Ro} \approx 2500$, and for molecular viscosity $\frac{1}{\text{Re}} \approx 10^{-12}$. We can conclude that the Coriolis term is much larger than either the non-linear advection or time derivative and that to the extent that time derivatives are important, they will overwhelm the non-linear term. Note that time scales shorter than a year will be even less influenced by the non-linear term, but that we need time scales of the order of a day before the time derivative term becomes comparable to the Coriolis term. We can also clearly exclude molecular viscous forces. We would need an eddy viscosity fifteen orders of magnitude larger than the molecular value for the viscous term to be comparable to the Coriolis term. Thus we seem completely justified in neglecting friction. I will leave it to you to convince yourself that reducing the length scale to 100 km does not substantially alter these conclusions.

We have just concluded that the Coriolis force must dominate the dynamics of the large scale ocean currents. The only remaining terms in the Navier-Stokes equation that can balance the Coriolis term are the pressure gradient and gravity. However, the horizontal components of the Coriolis term can never be balanced by gravity because it is vertical. Thus the horizontal Coriolis force must be balanced by the horizontal pressure gradient. This is called geostrophic balance. We have already noted that the vertical component of the Coriolis force is always very small compared to gravity, so that the balance in the vertical must be between gravity and the vertical pressure gradient. We can thus calculate the internal pressure field from the density distribution using the hydrostatic approximation. The horizontal gradient of this pressure can be used in the geostrophic balance to predict the velocity field. Figure 19(a) and (b) illustrate simple examples. This is called the dynamic method for determining currents.

Assume that the circulation is purely horizontal (a good approximation considering our extremely tiny estimate for the vertical flow necessary to maintain the permanent thermocline). In a locally flat right-handed coordinate system with $\hat{\mathbf{x}}$ east, $\hat{\mathbf{y}}$ north and $\hat{\mathbf{z}}$ up the components of the Navier-Stokes equation are

$$\hat{\mathbf{x}}: \qquad f\rho v = \frac{\partial p}{\partial x}$$

$$\hat{\mathbf{y}}: \qquad -f\rho u = \frac{\partial p}{\partial y}$$

$$\hat{\mathbf{z}}: \qquad -\rho g = \frac{\partial p}{\partial z}$$

where $f = 2\Omega \cos\theta$ is called the Coriolis parameter and θ is the colatitude. In writing the $\hat{\mathbf{z}}$ equation, we have dropped the very small vertical component of the Coriolis term (the Eotvos effect) relative to the gravity. We can eliminate p by differentiating the first equation with respect to z and the last with respect to x.

$$f \frac{\partial}{\partial z}(\rho v) = \frac{\partial^2 p}{\partial x \partial z} = -g \frac{\partial \rho}{\partial x}$$

We can likewise eliminate p using the z derivative of the second equation and the y derivative of the third.

$$-f \frac{\partial}{\partial z}(\rho u) = \frac{\partial^2 p}{\partial y \partial z} = -g \frac{\partial \rho}{\partial y}$$

These can be integrated vertically to give

$$u = -\frac{g}{\rho f} \int_{z_0}^{z} \left(\frac{\partial \rho}{\partial y}\right) dz + u(z_0)$$

and

$$v = \frac{g}{\rho f} \int_{z_0}^{z} (\frac{\partial \rho}{\partial x})\, dz + v(z_0)$$

The first term on the right is the velocity due to the internal mass distribution and is called the baroclinic velocity because it is related to the angle at which the surfaces of constant pressure are inclined relative to those of constant density. The integration constants $u(z_0)$ and $v(z_0)$ are independent of depth and are due to the topography of the ocean surface relative to the geoid (an equipotential of the gravity field). They are the components of the barotropic velocity and are independent of depth because the pressure gradient due to the surface tilt is independent of depth (see Figure 19(b)). Obviously a ship would have a difficult time determining the topography of the sea surface on the scales required. In principle, the ocean surface topography could be measured with a satellite equipped with a laser altimeter. However, no satellite presently exists for the purpose and the barotropic current remains a major uncertainty in the dynamical determination.

Oceanographers in the past have tried to estimate the integration constants by assuming that the current goes to zero at some depth (the so-called level of no motion). Modern measurements of deep currents have shown this assumption to be untenable. Some information can be extracted from the fact that the the baroclinic current sometimes does not satisfy conservation of mass (the continuity equation was not used in its derivation). However the amount of information is small and one still needs an additional assumption to make the barotropic current unique. This added assumption has typically constrained the smoothness or size of either the added velocity or the total velocity. Generally these constraints have little physical basis. Most of our present information about the barotropic velocity is deduced from the failure of the baroclinic velocity to equal the measured velocity. Large scale direct measurements of total transport with electromagnetic methods are likely to help significantly in the near future because they are directly sensitive to the vertically averaged velocity.

5.3. The Taylor-Proudman theorem

Geostrophic balance has some interesting consequences. One is the Taylor-Proudman theorem that says that when ρ is constant, the velocity cannot vary along the direction of the rotation axis. You can prove this rigorously by taking the curl of the equation of geostrophic balance.

$$\nabla \times 2\Omega \hat{\mathbf{k}} \times \mathbf{u} = -\frac{1}{\rho} \nabla \times \nabla p + \nabla \times \nabla \phi = 0$$

Using the vector identity

$$\nabla \times \mathbf{A} \times \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{A} - \mathbf{A} \cdot \nabla \mathbf{B} + \mathbf{A} \nabla \cdot \mathbf{B} - \mathbf{B} \nabla \cdot \mathbf{A}$$

and noting that $\nabla \cdot \mathbf{u} = 0$ and that $\hat{\mathbf{k}}$ is a constant vector and therefore has no divergence or gradient, the above result reduces to

$$(\hat{\mathbf{k}} \cdot \nabla)\mathbf{u} = 0$$

which is mathematical shorthand for the words of the theorem.

Figure 20 (and Figures 16.2 and 16.3 in Tritton) illustrates a remarkable consequence of the Taylor-Proudman theorem. Suppose you try to move a small block across the bottom of a rotating tank. The vertical velocity at the bottom of the tank and at the top of the block must be zero and the theorem implies that the vertical velocity must be zero everywhere. As the block moves, the fluid must move sideways to get around the block. The theorem implies that this horizontal velocity must also be independent of height and the fluid will move as if the the block extended the entire height of the tank. Exactly what happens to the fluid inside the so-called Taylor column directly above the block and the structure of the free shear layers at the outer vertical edge of the Taylor column are not described by the geostrophic equation because they involve other terms such as viscosity.

You might suppose that Taylor columns are not relevant to the oceans because the density is not constant. The equation of geostrophic balance can be written

$$2\Omega\hat{\mathbf{k}} \times \mathbf{u} = \frac{1}{\rho} \nabla p + \nabla\phi$$

If ρ is not constant, the curl of this equation becomes

$$2\Omega\nabla \times \hat{\mathbf{k}} \times \mathbf{u} = \nabla \frac{1}{\rho} \times \nabla p$$

which can be reduced to

$$\hat{\mathbf{k}} \cdot \mathbf{u} = \frac{-1}{2\Omega\rho^2} \nabla\rho \times \nabla p$$

This is the Taylor-Proudman theorem for a variable density fluid. The existence of Taylor columns requires that the term on the right is small. We note that

$$\nabla\rho = \nabla_H \rho + \frac{\partial\rho}{\partial z}\hat{\mathbf{z}}$$

and

$$\nabla p = \nabla_H p + \frac{\partial p}{\partial z}\hat{\mathbf{z}} = \nabla_H p + \rho g\hat{\mathbf{z}}$$

where ∇_H is the horizontal components of ∇. Thus

$$\nabla\rho \times \nabla p = \frac{\partial\rho}{\partial z}\hat{\mathbf{z}} \times \nabla_H p - \rho g\hat{\mathbf{z}} \times \nabla_H \rho$$

Now suppose that the flow is purely horizontal (i.e. $\mathbf{u} = \mathbf{u}_H$). Then we know that

$$2\rho\Omega\hat{\mathbf{k}} \times \mathbf{u}_H = -\nabla_H p$$

and the first term contributing to the right side of the Taylor-Proudman theorem is

$$\frac{-1}{2\Omega\rho^2}\frac{\partial\rho}{\partial z}\hat{\mathbf{z}} \times 2\rho\Omega\hat{\mathbf{k}} \times \mathbf{u}_H \approx \frac{u_H}{L}$$

where $L = \frac{1}{\rho}\frac{\partial\rho}{\partial z}$ is the scale length for the vertical variation of ρ. Since L is much larger than the depth of the ocean, this term is negligible. The surfaces of constant density in the ocean are typically tilted through a very small angle α with respect to horizontal and hence $|\nabla_H \rho| \approx \sin\alpha \frac{\partial\rho}{\partial z}$. Thus the magnitude of the second term contributing to the right side of the Taylor-Proudman theorem is

$$\frac{1}{2\Omega\rho^2}\rho g \sin\alpha \frac{\partial\rho}{\partial z} \approx \frac{1}{L}\frac{g \sin\alpha}{2\Omega}$$

Since $\frac{g \tan\alpha}{2\Omega}$ is the magnitude of the geostrophic velocity due to the tilt of the ocean-air interface, $\tan\alpha = \sin\alpha$ for small α and the geostrophic velocity associated with internal density variations is always much smaller than that due to the tilt of the upper surface, the second term is also negligible.

We conclude that the Taylor-Proudman theorem in the ocean takes the same form as in the uniform density fluid with the single caveat that the fluid velocity must be horizontal. Experimental work in rotating, strongly stratified fluids has demonstrated that Taylor columns still exist. However, they do not extend through the entire depth of the fluid, but are limited by the scale depth of the density variation as one might expect. While not important in the oceanic context, this limited length may be important in the deep planetary atmospheres of the gas giants such as Jupiter and in stars.

5.4. Modeling

Another important use of non-dimensional numbers comes from the fact that the equations governing the flow will be identical if the numbers are identical even though the values of the individual scales are very different. Thus we can model ocean currents in the laboratory by making Ω large and U small. We can then achieve a small Rossby number (dominant Coriolis term) with a modest value of L. We must, however, make sure when we do this that the Reynolds number remains very large or our experiment will have viscous effects not seen in the real ocean. For instance, suppose we want to demonstrate the Taylor column effect. It is not too difficult to make a rotating tank with $\Omega = 2\pi$ (1 revolution per second). Much faster than this and the plastic or glass outer sides may crack and fail. If we want the block to be 1 cm in diameter and the flow to be geostrophic, we require that

$$Ro = \frac{U}{2\Omega L} \approx \frac{U}{2\pi \cdot 0.01} \ll 1$$

which implies $U \ll 2\pi \approx 6$ cm/s. For the same length scale, The neglect of the viscous term requires

$$\mathrm{Re} = \frac{UL}{\nu} \approx \frac{U \cdot 0.01}{10^{-6}} \gg 1$$

which implies $U \gg 10^{-4}$ cm/s. There is a reasonably wide range of U which can satisfy both these conditions.

5.5. Angular momentum and vorticity

The physics behind the Taylor-Proudman theorem is essentially the conservation of angular momentum. If **u** varies in the direction of **Ω**, it will result in stretching or twisting of small small cylinders of fluid parallel to the rotation axis. If the velocity is perpendicular to the rotation axis and it varies along the rotation axis, it will cause the small cylinder to tilt and change the direction of **Ω** (see Figure 21(a)). I am sure you are familiar with a gyroscope and know that torque (rotational force) is required to twist the direction of the rotation axis. If the velocity varies along the rotation axis, it will stretch (or compress) the cylinder (see Figure 21(b)). If you stretch a cylinder of radius R_1 and length L_1 to a new length L_2, conservation of volume (i.e. mass when ρ is constant) implies that $\pi R_1^2 L_1 = \pi R_2^2 L_2$ and conservation of angular momentum implies that $R_1^2 \Omega_1 = R_2^2 \Omega_2$. Thus

$$\frac{\Omega_1}{\Omega_2} = \frac{L_1}{L_2} = \left[\frac{R_2}{R_1}\right]^2$$

Thus stretching the cylinder increases its rotation rate. The ratio of the rotational kinetic energies before and after is

$$\frac{KE_1}{KE_2} = \frac{R_1^2 \Omega_1^2}{R_2^2 \Omega_2^2} = \left[\frac{R_1}{R_2}\right]^2 \left[\frac{\Omega_1}{\Omega_2}\right]^2 = \frac{\Omega_2}{\Omega_1}\left[\frac{\Omega_1}{\Omega_2}\right]^2 = \frac{\Omega_1}{\Omega_2} = \frac{L_1}{L_2}$$

and stretching the cylinder increases its rotational kinetic energy. You can duplicate the physics of this effect by tying a small weight to the center of a piece of string. Then holding the ends of the string, spin the weight in a circle. When you pull on the string the weight will circle faster. Notice that you must exert a force to move your hands apart. In geostrophic balance, the kinetic energy of the fluid is very small compared to the rotational kinetic energy. (The Rossby number squared is the ratio of these energies.) Thus the fluid does not have enough energy to spontaneously do any stretching or twisting of the cylinders of fluid. The fluid therefore moves in such a way as to avoid stretching and twisting.

The quantity $\omega = \nabla \times \mathbf{u}$ is called the vorticity of the flow. The vorticity of uniform rotation, $\mathbf{u} = \Omega R\hat{\phi}$ is $2\Omega\hat{\mathbf{z}}$. However rotation is not the only way for a fluid to have vorticity. The vorticity of uniform shear (i.e. Couette flow) $\mathbf{u} = \frac{U}{D} z\hat{\mathbf{x}}$ is $\omega = \frac{U}{D}\hat{\mathbf{y}}$. All shear flows have vorticity and since shear and the effects of viscosity are almost inseparable, all viscous flows have vorticity. Vorticity can also be generated when gravity acts on a horizontal density gradient. The total gravitational force on a heavier blob of fluid will exceed that on a neighboring lighter element and the the fluid will twist. However, in the rest of this chapter, we will ignore gravitationally generated vorticity by assuming that ρ is constant. We will return to variable density later.

Using the vector identity

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \nabla\mathbf{B} + \mathbf{B} \cdot \nabla\mathbf{A} + \mathbf{A} \times \nabla \times \mathbf{B} + \mathbf{B} \times \nabla \times \mathbf{A}$$

with $\mathbf{A} = \mathbf{B} = \mathbf{u}$, we can express the non-linear advection term in the Navier-Stokes equation in its invariant form

$$\mathbf{u} \cdot \nabla\mathbf{u} = \omega \times \mathbf{u} + \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u})$$

which involves vector differential operators that do not require spatial derivatives of the unit vectors and is thus quite useful for figuring out the correct form of this term in curvilinear coordinate systems. We also see that it involves the Coriolis force due to the vorticity plus the gradient of the the kinetic energy per unit mass. This latter is called the dynamic pressure for obvious reasons.

Substituting the above into the Navier-Stokes equation and taking its curl, we obtain the vorticity equation

$$\frac{\partial\omega}{\partial t} + \mathbf{u} \cdot \nabla\omega = \omega \cdot \nabla\mathbf{u} + \nu\nabla^2\omega$$

where we have made use of the vector identity given earlier to conclude that

$$\nabla \times \omega \times \mathbf{u} = \mathbf{u} \cdot \nabla\omega - \omega \cdot \nabla\mathbf{u} + \omega(\nabla \cdot u) - \mathbf{u}(\nabla \cdot \omega)$$

Note that both $\nabla \cdot \omega$ and $\nabla \cdot \mathbf{u}$ are zero. The left side of the vorticity equation is clearly $\frac{D\omega}{Dt}$, the rate of change of vorticity seen by an observer moving with the fluid. The term $\omega \cdot \nabla\mathbf{u}$ on the right has a simple physical interpretation illustrated in Figure 22. A line which is everywhere parallel to ω is called a vortex line. When $\mathbf{u}$ is parallel to ω and increases in the direction of ω, it will stretch the vortex lines. We know from our earlier discussion that such stretching will increase the local rotation rate. On the other hand, if $\mathbf{u}$

is perpendicular to ω and varies along a vortex line, it will tilt (or twist) the vortex line and hence change the direction of ω. Thus $\omega \cdot \nabla \mathbf{u}$ is the change in vorticity due to stretching or twisting of vortex lines by the velocity field.

The final term on the left expresses what we already know: viscosity can result in shear which implies vorticity. We also see that viscosity can be thought of as a diffusion coefficient for vorticity as well as momentum. If $\nu = 0$, the vorticity equation reduces to the mathematical expression of the Helmholtz vorticity theorem

$$\frac{D\omega}{Dt} = \omega \cdot \nabla \mathbf{u}$$

which states that in the absence of density variations or viscosity, vorticity can only be changed by stretching or twisting of the vortex lines. This theorem has two interesting corollaries:

(1) If the vorticity of a constant density inviscid fluid is zero at any instant it subsequently will always be zero.

(2) A two-dimensional flow of a constant density inviscid fluid must have constant vorticity.

The first follows from the fact that if $\omega = 0$, the time derivative of ω becomes zero and therefore the vorticity must remain zero. The second is true because the vorticity of a two-dimensional flow is perpendicular to the plane of the flow and thus the $\omega \cdot \nabla \mathbf{u}$ is always zero.

5.6. The velocity potential and the stream function

A flow without vorticity is called irrotational. Any vector field $\mathbf{u} = \nabla \chi$ is automatically irrotational. The scalar χ is called the velocity potential. If the fluid is also incompressible

$$\nabla \cdot \mathbf{u} = \nabla \cdot \nabla \chi = \nabla^2 \chi = 0$$

Thus the velocity potential satisfies the Laplace equation. This equation is known to have a unique solution in three cases:

(1) The gradient of χ perpendicular to a boundary enclosing the fluid is known. This is called a Neumann boundary condition. It is obvious from the definition of the velocity potential that this implies knowing the velocity component perpendicular to the boundary.

(2) The value of χ is known on the closed boundary. This is called a Dirichlet boundary condition. Since knowing χ permits calculation of its derivative parallel to the boundary, this implies knowing the velocity components tangential to the boundary.

(3) A linear combination of the the last two conditions. This is called a Cauchy boundary condition.

The important point from a physical point of view is that one cannot specify both the normal and tangential velocity at the boundary of an irrotational, incompressible fluid. Since irrotational is essentially synonymous with an inviscid fluid (a viscous fluid can never be everywhere irrotational if there are boundaries), the practical consequence is that inviscid flow can rarely satisfy the requirement that all components of the velocity go to zero at a rigid boundary. Another way to look at this is to note that dropping the viscous term in the Navier-Stokes equation reduces the order of the differential equation and thus the number of boundary conditions that can be simultaneously satisfied. Thus in the real world, high Reynolds number flows (which are not necessarily irrotational) cannot satisfy the physically necessary boundary conditions at a rigid boundary. This problem is dealt with in thin transition layers called boundary layers in which viscosity is important and the the full boundary conditions can be met.

When flow is two-dimensional, the velocity components

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

automatically satisfy the incompressibility condition $\nabla \cdot \mathbf{u} = 0$. The scalar ψ is called the stream function. Note that its sign is arbitrary. One can show that the lines of constant ψ are everywhere parallel to **u** and are called streamlines. For steady flow, the streamlines are the paths of fluid particles, but this may not be true for time varying flow. Unlike the velocity potential, utility of the stream function is not limited to irrotational flow. The vorticity is easily shown to be

$$\omega = -\nabla^2 \psi$$

If the flow is irrotational, however, ψ also satisfies the Laplace equation and the lines of constant ψ and χ are orthogonal to each other. We also must have that

$$u = \frac{\partial \chi}{\partial x} = \frac{\partial \psi}{\partial y}$$

and

$$v = \frac{\partial \chi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

If the two-dimensional velocity field is replaced with the complex function $U = u + iv$, the above relations are known as the Cauchy-Reiman conditions and they imply that U is an analytic function. This opens up the very powerful tools of conformal mapping for finding solutions for complicated boundary geometry. However, since two-dimensional, irrotational flow is of little interest in this course we will not consider such techniques further.

5.7. Bernoulli's theorem

The invariant form of the Navier-Stokes equation for a constant density fluid can be written

$$\frac{\partial \mathbf{u}}{\partial t} + \omega \times \mathbf{u} = \nabla(\phi + \frac{p}{\rho} + \frac{1}{2}\mathbf{u} \cdot \mathbf{u}) + \nu \nabla^2 \mathbf{u}$$

For steady, inviscid flow, this becomes

$$\omega \times \mathbf{u} = \nabla(\phi + \frac{p}{\rho} + \frac{1}{2}\mathbf{u} \cdot \mathbf{u})$$

The vector $\omega \times \mathbf{u}$ is perpendicular to both ω and $\mathbf{u}$. Thus the gradient of the quantity $H = \phi + \frac{p}{\rho} + \frac{1}{2}\mathbf{u} \cdot \mathbf{u}$ is also perpendicular to both $\mathbf{u}$ and ω and we can conclude that H must be constant along both stream lines and vortex lines. This result is known as Bernoulli's theorem. Note that H is the sum of three terms related to fluid energy and is the Hamiltonian of the flow.

When the flow is irrotational, we have a special form of Bernoulli's theorem

$$\phi + \frac{p}{\rho} + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} = constant$$

An immediate consequence is that the non-hydrostatic pressure varies inversely with velocity. Where the velocity is high the pressure must be low and *vice versa*. This has variety of important predictions for high Reynolds number flow. For instance the pressure above an airfoil will be lower than below resulting in lift; the pressure inside a constriction in a pipe will be lower (the Venturi effect) and the pressure above the highs of a corrugated surface will be lower than in the troughs. If the corrugation is a water wave, this pressure field would cause the wave to grow*. These examples are all illustrated in Figure 23.

A final example shown in Figure 24 is the adverse pressure gradient in the downstream part of the flow near a cylinder. This pressure gradient can actually reverse the flow in the boundary layer near the cylinder and result in the phenomenon known as flow separation.

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*A more important mechanism for causing a surface wave to grow is the wave-like pressure distribution associated with shear flow instability within the boundary layer in the air above the surface. Shear flows (including Couette and Poiseuille flow) are notoriously unstable. The generic shear flow instability is the Kelvin-Helmholtz instability described in Tritton, section 16.7.

5.8. The Ekman boundary layer

Although geostrophic balance is observed to hold very accurately in the oceans (and the atmosphere), it cannot be the entire story. For one thing, the forces are perpendicular to the motion, so that they cannot be responsible for getting the motion going. Furthermore, there is ultimately a physical requirement that the velocity go to zero when it meets a horizontal boundary. At some small scale near the boundary, viscosity must become important. The boundary layer that forms at the top and bottom of the ocean can actually be observed at the bottom of a tea cup which has been stirred (see Figure 25). Tea leaves on the bottom spiral in towards the center and gather in a pile. This is contrary to one's intuition that particles heavier than water ought to be thrown outwards by the centrifugal force. One view of the physics is that the circular motion of the water in the cup requires a balance between centrifugal force and a pressure gradient force caused by a tilt of the top surface. Since the required force (which is proportional to the slope of the surface) must increase linearly with distance from the rotation axis, the shape of the top surface is parabolic. This pressure gradient force is uniform throughout the depth range. Near the bottom boundary, however, viscous effects slow the fluid so that the centrifugal force is no longer sufficient to balance the pressure gradient force. There is therefore a net force which pushes the tea leaves towards the center. Since the water near the boundary is still rotating (albeit slower than the water above) the tea leaves follow a spiral path to the center. What would an observer moving with the rotating fluid see? (Answer: Since he is going around the center faster than the fluid near the boundary, he would see the tea leaves spiral towards the center but the sense of the spiral would be reversed.)

An important consequence of the radial flow near the boundary is that the water stops rotating much more quickly than it would if viscous diffusion had to act over the radius or depth of the cup. For a cup with a 5 cm radius, the time scale estimated by the methods of the previous chapter would be $\tau = \frac{R^2}{\nu} \approx 2500$ seconds. You would expect it to take almost an hour for the tea to come to rest. However, you can easily verify by observation that it stops in more like 100 seconds. This is primarily due to continually pumping fluid through the boundary layer where viscosity can act. If you look very closely, you may see that some of the smallest leaves get lifted up right at the center due to the upflow required by the converging flow in the boundary layer. If this happens you will see them suddenly be flung radially outwards by the centrifugal force when they get above the region where viscosity is important.

This boundary layer can be thought of as a velocity which is zero at some height above the boundary and reaches its maximum at the boundary, where it is equal in magnitude and opposite in direction to the basic rotation. This is the velocity that the observer discussed above fixed to the basic rotation would see. From his point of view, the tendency of the tea leaf to go towards the center is the action of a Coriolis force due to the motion of the boundary relative to the fluid far away from the boundary. He would conclude that the basic balance of force in the boundary layer is between this Coriolis force and the viscous force associated with the velocity shear near the boundary.

We shall make this more precise by considering the boundary layer at the surface of a deep ocean at rest acted on by a steady wind stress $\tau_{xz} = \mu S \hat{\mathbf{x}}$. Since the main stream velocity is assumed to be zero, there is no horizontal pressure gradient in either the main stream or the boundary layer. Furthermore, since the flow in the boundary layer will be horizontal and vary only in the $\hat{\mathbf{z}}$ direction, the non-linear advection term will also be zero. The Navier-Stokes equation reduces to

$$2\Omega \times \mathbf{u} + \frac{1}{\rho}\frac{\partial p}{\partial z}\hat{\mathbf{z}} = -g\hat{\mathbf{z}} + \nu\nabla^2\mathbf{u}$$

Assuming that the boundary layer is thin and that we can ignore the horizontal relative to vertical gradients (i.e the usual boundary layer assumptions). the components at colatitude θ become

$$\hat{\mathbf{x}}: \qquad -fv = \nu\frac{d^2u}{dz^2}$$

$$\hat{\mathbf{y}}: \qquad fu = \nu\frac{d^2v}{dz^2}$$

$$\hat{\mathbf{z}}: \qquad \frac{\partial p}{\partial z} = -\rho g$$

where $f = 2\Omega\cos\theta$ is the Coriolis parameter. Note that the first two equations inextricably couple the two horizontal velocity components together. Neither can be zero without the other also being zero. Thus the boundary layer flow is fundamentally two dimensional. The third equation is decoupled from the other two and simply states that the fluid pressure is hydrostatic. We want to find a solution to the first two equations subject to the boundary conditions:

$$@z = 0: \qquad \frac{du}{dz} = S \qquad \frac{dv}{dz} = 0$$

$$@z \rightarrow \mathit{infinity}: \qquad u = v \rightarrow 0$$

A trick which simplifies the algebra is to let $U = u + iv$. If we multiply the $\hat{\mathbf{y}}$ equation by i and add it to the $\hat{\mathbf{x}}$ equation we obtain the single equation

$$ifU = \nu\frac{d^2U}{dz^2}$$

which has a general solution

$$U = Ae^{\alpha}z$$

where

$$\alpha = \pm \sqrt{\frac{if}{\nu}} = \pm \sqrt{\frac{f}{2\nu}}\,(1+i) = \pm \frac{(1+i)}{\delta}$$

and $\delta = \sqrt{\frac{2\nu}{f}} = \sqrt{\frac{\nu}{\Omega}\cos\theta}$ is the Ekman thickness that we met earlier in considering the cone-plate viscometer., To satisfy the boundary conditions as $z \rightarrow -\,infinity$ we must clearly take only the solution with $Re[\alpha] > 0$. The two boundary conditions at $z = 0$ become $\frac{dU}{dz} = S$ (because S is real). Therefore

$$A = \frac{S\delta}{1+i} = \frac{S\delta(1-i)}{2}$$

and finally

$$U = \frac{S\delta}{1+i}\, e^{(1+i)\frac{z}{\delta}}$$

The velocity components are

$$u = Re[U] = \frac{S\delta}{\sqrt{2}}\, e^{\frac{z}{\delta}} \cos\left(\frac{z}{\delta} - \frac{\pi}{4}\right)$$

and

$$v = Im[U] = \frac{S\delta}{\sqrt{2}}\, e^{\frac{z}{\delta}} \sin\left(\frac{z}{\delta} - \frac{\pi}{4}\right)$$

We have thus found a boundary layer solution for the rotating case which is quite different than in the absence of rotation. The non-rotating (Blasius) boundary layer grows with time and would effect the entire ocean over geologic time. The rotating boundary layer has a constant thickness. At 45^o latitude, $\delta = 10$ cm for the molecular viscosity of water. Increasing the viscosity by a factor of 1000 to to account for turbulent eddy diffusion increases δ to only 3 meters. Obviously the ocean does not have to be very deep for the above theory assuming infinite depth to be correct! We can also conclude that boundary friction will have little influence on the interior of the ocean and that an assumption of geostrophic balance in the interior is quite reasonable.

In a tea cup with $\Omega = 1$ radian per second and molecular viscosity, $\delta \approx 1$ mm, which is much less than the radius of the cup as required. The atmospheric layer can be thicker than in the ocean (about 1 km) due to the more vigorous mixing associated with thermals and the shedding of eddies from surface roughness.

Taking the ratio of the velocity components at the surface we obtain

$$\frac{u}{v} = \frac{\cos(\frac{-\pi}{4})}{\sin(\frac{-\pi}{4})} = -1$$

Thus the surface velocity in the Northern Hemisphere is rotated 45^o to the right with respect to the surface shear stress. Intuitively you can think of the boundary applying applying shear stress to the adjacent fluid which is then deflected by the Coriolis force. The deflected flow applies shear stress to fluid further from the boundary, which is deflected even more by the Coriolis force. The result is that the direction of **u** spirals as its magnitude decays away from the boundary (see Figure 26(a)). If we integrate the complex velocity U from the surface to a depth sufficient that the velocity has effectively fallen to zero, we obtain

$$\bar{U} = \int_{-D}^{0} \frac{S\delta}{1+i} e^{(1+i)\frac{z}{\delta}} \, dz = -\frac{S\delta^2 i}{2}$$

Since this is negative and purely imaginary, it is in the negative $\hat{\mathbf{y}}$ direction. Thus the mass transport by the boundary layer is rotated clockwise 90^o with respect to the surface stress and 45^o with respect to the surface velocity.

Observations of the discrepancy between the drift of thin sea ice and the wind direction were reported by Nansen and investigated by Ekman. This boundary layer is therefore called the Ekman layer, and the spiral with depth the Ekman spiral. What one actually see depends on whether one views the velocity from a frame fixed to the boundary or fixed to the main stream far away from the boundary. At the surface of an ocean acted on by surface shear stress due to the wind, it is appropriate to think of the deep water as having no velocity (see Figure 26(b)). On the other hand, for the teacup experiment or watching pollution drifting from a smokestack you are viewing the Ekman layer from a frame moving with the boundary. Since the main stream velocity is much larger than the boundary layer velocity except right at the boundary, the flow does not reverse at any depth. The total velocity simple swings as it moves away from the boundary and then oscillates around the direction of the main stream (see Figure 26(c)).

Although both the atmosphere and ocean have boundary layers which look strikingly like the classical Ekman layer just described. The actual situation is more complicated. One of the complications is that the Ekman layer is weakly unstable. This is discussed in more detail by Tritton in Section 16.5. The instabilities are the result of the curvature of the shear and are quite unusual because their growth rate is only polynomial in time rather than exponential. Thus the instabilities do not run away in a catastrophic manner and do not destroy the main structure of the layer. Organized lines of clouds sometimes seen within the atmospheric pseudo-Ekman layer are attributed to these weak instabilities.

5.9. Ocean Currents

We now have all the pieces necessary to explain the basic features of the general circulation of the oceans.

5.9.1. Average wind stress over the oceans

A thorough discussion of the general circulation of the atmosphere is beyond the scope of this course (it will be considered in more detail in Geophysics 406). However, since the ultimate source of energy for the surface currents in the oceans is the Sun, and its thermal energy is transferred to kinetic energy in the ocean through the action of the winds on the surface, we do need to say something about this process.

The main fact is that the incident solar energy exceeds infra-red radiation to space in the tropics, while the opposite is true near the poles. There is therefore a net rising of light warm air near the equator and sinking of cold dense air near the poles. One might therefore expect a single Hadley cell in each hemisphere with a net equator-ward flow. This flow would be influenced by Coriolis forces so that the flow would gain a westward component (see Figure 27(a)) and one would expect to see easterly winds at all latitudes. In actuality the Hadley cells are subject to a variety of instabilities. One of most significant breaks each cell into three globe-encircling cells (and an even larger number of cells on the gas giants such as Jupiter; see Figure 27(b)). (The surface flow is still equator-ward near the poles and the equator and one sees polar easterlies at high latitudes and the easterly trade winds at low latitudes. At mid-latitudes, however, the surface flow is pole-ward and one sees predominantly westerly winds.

Most of the world's oceans do not extend into the region influenced by the polar easterlies, thus the wind stress on the North Atlantic or Pacific Oceans is on average a clockwise torque. The trades blow from east to west in the lower latitudes and the westerlies blow from west to east at mid-latitudes. Since the continents block circulation of water along lines of latitude around the globe, the surface circulation due to the clockwise torque is likely to be a clockwise gyre. This surface current will be limited in depth extent because of the Ekman layer.

The situation in the Antarctic is different. There the ocean does circle the globe and is acted on primarily by the westerlies. A slow (10 cm/s) globe-circling current parallel to the wind is observed. One can imagine northward transport in the Ekman layer building up a tilt of the sea surface (down to the south). This tilt produces a pressure gradient force that is the same at all depths and a hence a geostrophic current that is very deep and primarily barotropic. The great depth extent of the Antarctic Circumpolar Current means it carries more volume than any other current.

5.9.2. Stommel's theory of westward intensification

We might expect the surface torque to accelerate the surface water to speeds comparable to the average atmospheric wind speeds. The fact that this is not observed (1 m/s is a fast ocean current while wind speeds exceeding 10 m/s are common) indicates that some form of frictional force is partially balancing the wind torque. Because the Ekman layer isolates the friction applied to the surface from the deep water, friction can only act from the side. Clearly molecular diffusion and even the enhanced vertical diffusion discussed in the context of the seasonal thermocline are too slow to be useful in this context. However, satellite images of sea surface temperature and color clearly demonstrate the existence of oceanic eddies (particularly near boundaries) whose scales go up to hundreds of km. (They can be big enough to be geostrophic themselves and in many respects resemble weather systems in the atmosphere.) Although a complete theory of horizontal mixing does not yet exist for the ocean, it almost certainly can be approximated by a theory with a diffusivity orders of magnitude larger than those already discussed. For our purposes it is only necessary to convince ourselves that frictional forces can act from the side. These friction forces must oppose the wind forcing and thus necessarily be a counter-clockwise torque in the Northern Hemisphere.

The wind torque is fairly uniform and in the absence of some other kind of physics, one might expect uniform viscous dissipation and hence uniform velocity gradients. This would imply a uniform gyre. Specifically, we would not expect a profound asymmetry in the structure of the northward currents in the west and the southward currents in the east. However, as we have already noted in Figure 18, observation shows that the western and eastern boundaries are profoundly different with a narrow, warm current in the west and a much slower cold current in the east. Henry Stommel (probably the only full professor in a Harvard science department in modern times without a Ph.D.) developed a simple theory that can account for this asymmetry.

He started, by modifying the Ekman layer equations to include the possibility of frictional influence from the sides. The horizontal components become

$$\hat{\mathbf{x}}: \qquad -\rho f v + \frac{\partial p}{\partial x} = \frac{\partial}{\partial z}\,\mu_V\,\frac{\partial u}{\partial z} + \frac{\partial}{\partial y}\,\mu_H\,\frac{\partial u}{\partial y}$$

$$\hat{\mathbf{y}}: \qquad \rho f u + \frac{\partial p}{\partial y} = \frac{\partial}{\partial z}\,\mu_V\,\frac{\partial v}{\partial z} + \frac{\partial}{\partial x}\,\mu_H\,\frac{\partial v}{\partial x}$$

where he included only the horizontal stresses associated with gradients perpendicular to the velocity. This is clearly unreasonable in corners but reasonable where the flow is parallel to a boundary. Then he averaged these equations vertically over a distance D much larger than the Ekman thickness. Term by term we get

$$\int_{-D}^{0} \rho f u \; dz \equiv fM_x \qquad\qquad \int_{-D}^{0} \rho f y \; dz \equiv fM_y$$

$$\int_{-D}^{0} p \; dz \equiv P$$

$$\int_{-D}^{0} \frac{\partial}{\partial z}\, \mu_V \frac{\partial u}{\partial z} = \mu_V \frac{\partial u}{\partial z}\Big|_{z=0} \equiv T_x \qquad \int_{-D}^{0} \frac{\partial}{\partial z}\, \mu_V \frac{\partial v}{\partial z} = \mu_V \frac{\partial v}{\partial z}\Big|_{z=0} \equiv T_y$$

$$\int_{-D}^{0} \frac{\partial}{\partial x}\, \mu_V \frac{\partial v}{\partial x} \equiv F_x \qquad \int_{-D}^{0} \frac{\partial}{\partial y}\, \mu_V \frac{\partial u}{\partial y} \equiv F_y$$

where M_x and M_y are the components of mass transport per unit width, T_x and T_y are the components of the wind stress and F_x and F_y are the components of the shear stresses acting from the sides. The vertically averaged equations are

$$-fM_y + \frac{\partial P}{\partial x} = T_x + F_x$$

$$fM_x + \frac{\partial P}{\partial y} = T_y + F_y$$

The averaged geostrophic pressure can be eliminated by cross-differentiating these two equations and subtracting giving

$$f\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y}\right) + \frac{\partial f}{\partial y}\, M_y = \left(\frac{\partial T_y}{\partial x} - \frac{\partial T_y}{\partial y}\right) + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_y}{\partial y}\right)$$

where we have included the possibility that the Coriolis parameter may depend on latitude (although it does not depend on longitude). The vertical averaged conservation of mass equation becomes

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} = 0$$

Thus we can reduce the above equation to

$$\frac{\partial f}{\partial y}\, M_y = \tau + \eta$$

where we have defined

$$\tau \equiv \frac{\partial T_y}{\partial x} - \frac{\partial T_y}{\partial y} \qquad \eta \equiv \frac{\partial F_y}{\partial x} - \frac{\partial F_y}{\partial y}$$

This equation is actually the vertically averaged $\hat{\mathbf{z}}$ component of the vorticity equation (cross-differentiation and subtraction is equivalent to taking the curl). Each term has a simple physical meaning: τ is the torque applied to the top surface by the wind and by the right hand rule is negative in the Northern Hemisphere. η is the torque applied to the side by the friction. It will always be positive because it basically balances τ. In fact, in the simplest approximation in which we assume $\frac{\partial f}{\partial y} = 0$, we have a strict balance of the two torques and the uniformity of the wind torque implies a uniform gyre with no western intensification. The final term represents the conservation of angular momentum for vortex tubes (i.e. Taylor columns). Since $\frac{\partial f}{\partial y} > 0$ in the Northern Hemisphere, this angular momentum term will always be positive on the western side of a Northern Hemisphere ocean basin (where the flow is northward and hence $M_y > 0$) and negative on the eastern side. Since τ is fixed, the averaged vorticity equation requires that η be larger on the west than the east. Larger friction implies larger velocity gradients. Finally, since the mass flux north in the west must equal the mass flux south in the east, we are forced to conclude that the western boundary current must be narrower and more intense than the eastern boundary current.

One can come to the same conclusion from purely physical considerations. Taylor columns in the ocean are parallel to the rotation axis and therefore not vertical. They will be shortened as they move north in the Northern Hemisphere ocean (see Figure 28(a)) and stretched as they move south (except in very narrow region around the equator). Those moving north develop a clockwise rotation relative their surroundings, those moving south develop a counterclockwise rotation (see Figure 28(b)). The northward moving columns appear to be under the influence of a torque which is in the same sense as the wind torque and the balance of torques on the west side of the ocean requires an enhanced frictional torque. The southward moving columns experience a torque opposite to the wind torque and thus the east side of an ocean requires less frictional torque to keep the forces in balance.

Figure 29(a) shows Stommel's result for the simplified case of a constant uniform wind torque and $f = f_0 + \beta y$. Figure 29(b) shows a calculation by Munk using a more realistic latitudinal dependence of the wind torque. The polar easterlies generate a second clockwise gyre with western intensification in the northern part of the ocean that is represented by the Labrador Current in the Atlantic ocean. The equator-ward flow declines as one approaches the equator as the flow is converted to net upward flux, thus the trades have a maximum north and south of the equator and the region of of very low winds near the equator is called the doldrums. The minimum (but not reversal) of the wind stress near the equator results in clockwise torque and another gyre entirely confined to the region near the Equator. This gyre is also observed and its components North of the equator are called the North Equatorial Current and the North Equatorial Counter Current.

5.9.3. Deep Currents

We have already discussed deep currents in the context of the maintenance of the permanent thermocline. However, because their scale is large they are also affected by Coriolis forces and are not simply a uniform equator-ward flow. There are only two sources of deep water: off Greenland and in the Wedell Sea of Antarctica both during local winter. In each case, the dense water flowing towards the equator curves to the western side of the ocean basin. Figure 30(a) shows a simplified east-west cross-section of the South Atlantic. The top surface of the northward flowing current tilts down to the east in accordance with geostrophic balance. As the deep western water flows north, the Coriolis force weakens so that there is an excess pressure gradient to drive a flow eastward normal to the boundary. If this did not happen, the current would eventually meet the southward flowing current from Greenland and cause a pile-up at the equator which would result in a pole-ward pressure gradient in both hemispheres. This pressure gradient would be in geostrophic equilibrium with an eastward current and one again predicts outflow from the western boundary current. The outward flow curves pole-ward. If the flow curved equator-ward as one might expect from consideration of the direct effect of Coriolis force, it would pile up dense water at the equator resulting in a pole-ward pressure gradient that would strengthen the eastward geostrophic flow. If the flow continued directly across the ocean basin it would pile up dense water on the eastern side resulting in a westward pressure gradient which would be in geostrophic equilibrium with pole-ward flow. It is this pole-ward flow that is observed. The approximate pattern of the deep flow is shown in Figure 30(b). Remember that this flow does not close on itself, but instead continually diminishes due to the net upward flux. The actual pattern is disturbed by the blocking effect of the mid-ocean ridges. This blocking is not absolute because the deep currents extend to depths above the tops of the ridges (see Figure 30(c) and there are gaps in the ridges at fracture zones. These deep currents are orders of magnitude slower than tidal flows and so cannot be measured directly. However, tidal flows have no net transport when averaged over the tidal period. Thus the pattern of the very slow deep circulation has been verified using methods such as the decay of *carbon*^{14} to measure the time since the water was last in contact with the atmosphere. The Pacific differs somewhat from the Atlantic because there is no source of deep water in the North Pacific.

Finally there are subsurface currents coincident with the the equator. Although very weak or non-existence at the surface, they have a strong maximum near the largest vertical density gradient (about 100 meters). They extend approximately two degrees north and south of the equator and flow from east to west. These currents were first observed in the Pacific where it is called the Cromwell Current in honor of its discoverer who died shortly thereafter. They have volume transports comparable to the Gulf Stream and must be considered among the oceans' major flows. Nevertheless their origin remains controversial with non-linear rectification of large scale, quasi-geostrophic waves at the singularity where the Coriolis force goes to zero at the equator among the leading candidates.