# Notes on a catastrophe: a feedback analysis of Snowball Earth

Gerard H. Roe and Marcia B. Baker

*Department of Earth and Space Sciences,*

*University of Washington, Seattle, WA.*

February 25, 2010

# Abstract

**The language of feedbacks is ubiquitous in contemporary Earth Sciences, and the framework of feedback analysis is a powerful tool for diagnosing the relative strengths of the myriad mutual interactions that occur in complex dynamical systems. The ice albedo feedback is widely taught as the classic example of a climate feedback. Moreover, its potential to initiate a collapse to a completely glaciated Snowball Earth is widely taught as the classic example of a climate 'tipping point'. A feedback analysis of the Snowball Earth phenomenon in simple, zonal-mean energy balance models clearly reveals the physics of the snowball instability and its dependence on climate parameters. The analysis can also be used to illustrate some fundamental properties of climate feedbacks: how feedback strength changes as a function of mean climate state; how small changes in individual feedbacks can cause large changes in the system sensitivity; and finally, how the strength and even sign of the feedback is dependent on the climate variable in question.**

# 1 Introduction

Early efforts to represent Earth's climate with energy balance models uncovered the disconcerting possibility that a relatively small decrease in the solar output might lead to a catastrophic global glaciation - the result of a runaway ice-albedo feedback (e.g., Budyko, 1969; North, 1975; Lindzen and Farrell, 1977). Although the issue remains controversial (e.g., Fairchild and Kennedy, 2007; Allen and Etienne, 2008) assorted lines of geological evidence appear to indicate that Earth passed through several episodes of complete, or near-complete, glaciation during the Proterozoic (e.g., Kirschvink, 1992; Hoffman et al., 1998; Hoffman and Li, 2008). Follow-up integrations of more-comprehensive global climate models have also found climate states with a global or near-global glaciation, though they typically require larger reductions in the solar output than the earlier calculations suggested (e.g., Baum and Crowley, 1993; Jenkins and Smith, 1999; Crowley et al., 2001).

To our knowledge, the factors controlling Snowball Earth have never been presented in terms of a formal feedback analysis, and doing so provides an opportunity to demonstrate several basic properties of feedbacks. Applying this analysis to the original zonal-mean energy balance climate models, the physical mechanism of the runaway glaciation can be clearly and simply demonstrated. The strength of the feedback is shown to equal the ratio of competing stabilizing and destabilizing tendencies on the global energy balance or, equivalently, competing tendencies on the local energy budget at the advancing ice-line. The phenomenon of a snowball Earth is a simple illustration of how climate sensitivity and feedback strength can change as a function of the mean climate state, which is an issue of some relevance for future climate predictions. Moreover, although there are obvious caveats because of the simplifying assumptions of the models, the instability is also an interesting example of a climate ‘tipping point’.

The analytical solutions for the simple energy balance models permit feedback strengths to be calculated even for the unstable equilibrium climates. Doing so gives the somewhat counterintuitive but explainable result, that the ice-albedo can under some conditions behave as a negative feedback on global mean temperature. The cause is the peculiar physics of the small ice-cap instability (e.g., North, 1975), and that of a previously unreported counterpart at low latitudes.

## 2 Analysis

We begin with the classic equation for the annual-mean, zonal-mean energy balance model (EBM) as a function of latitude (e.g., Budyko, 1969; North, 1975; Lindzen, 1990):

$$\frac{Q}{4}S(x)(1-\alpha(x)) = A + BT(x) + \nabla \cdot \vec{F}. \tag{1}$$

where $Q$ is the solar constant and $x$ is sine of latitude. $T(x)$, $S(x)$ and $\alpha(x)$ are the local temperature, normalized annual-mean insolation and the albedo, respectively. $A + BT(x)$ is a linearization of the outgoing longwave radiation (OLR), and $\vec{F}$ is the poleward heat transport.

Equation (1) can be integrated from equator to pole to give an expression for the global energy balance:

$$\frac{Q}{4}(1-\alpha_p) = A + B\overline{T}, \tag{2}$$

where $\alpha_p$ is the global-average albedo:

$$\alpha_p \equiv \int_0^1 \alpha(x)S(x)dx. \tag{3}$$

Finally, let $x_s$ be the latitude of the ice-line (i.e., where $T = T_s$).

To a good approximation $S(x)$ may be represented as $S(x) = 1 + s_2P_2(x)$, where $s_2 = -0.482$ and $P_2$ is the second Legendre polynomial: $P_2 = \frac{1}{2}(3x^2 - 1)$ (e.g., Chylek and Coakley (1975), Figure 1a). We adopt parameter values from Lindzen and Farrell (1977): $A = 211.1$ W m$^{-2}$; $B = 1.55$ W m$^{-2}$ $^o$C$^{-1}$. Note that the unit of $T$ is $^o$C. We allow $Q$ to vary in the vicinity of the modern day value, which Lindzen and Farrell took to be $Q_0 = 1336$ W m$^{-2}$.

If $\alpha_p =$ constant, $x_s$ and $\overline{T}$ respond directly (with no feedbacks) to variations in $Q$. A feedback can be introduced by allowing albedo to be a function of temperature: an ice-free albedo, $\alpha_1$, is assumed for temperatures greater than $T_s$ (typically -10 $^o$C), and an ice-covered albedo, $\alpha_2$, is assumed for temperatures less than $T_s$. Following Lindzen and Farrell (1977) we take $\alpha_1 = 0.3, \alpha_2 = 0.6$. Therefore, from equation (3)

$$\alpha_p(\overline{T}) = \alpha_p(x_s(\overline{T})) = \alpha_1 \int_0^{x_s} S(x)dx + \alpha_2 \int_{x_s}^1 S(x)dx. \tag{4}$$

Using the relationship between Legendre polynomials that $(2n+1)P_n(x) = \frac{d}{dx}[P_{n+1}(x) - P_n(x)]$ (e.g., Abramovitz and Stegun,1965), equation (4) can be written as:

$$\alpha_p(x_s) = \alpha_2 + (\alpha_1 - \alpha_2)\left[x_s + \frac{s_2}{5}(P_3(x_s) - P_1(x_s)\right]. \tag{5}$$

Figure 1b shows that $\alpha_p(x_s)$ varies smoothly between the ice-free and ice-covered limits.

## 2.1 Budyko-style energy balance models

Budyko (1969) presented an energy balance model that is particularly tractable analytically, proposing a very simple parameterization for the divergence of the poleward heat flux:

$$\nabla \cdot \vec{F} = C(T - \overline{T}), \tag{6}$$

where the overbar denotes the global mean. Thus there is a divergence of heat flux if the local temperature is higher than the global mean, and convergence of heat flux if it is lower. The higher the value of $C$, the more efficiently heat is redistributed on the planet. Sellers (1969) also parameterized heat flux in this way, but included extra model complexities that are unnecessary for present purposes.

#### 2.1.1 Traditional Analysis

An outline of the solution is briefly given here for clarity of presentation, but follows previous studies (e.g., Lindzen and Farrell, 1977).

With this Budyko-style parameterization of the heat flux, applying equation (1) at the ice-line ($x = x_s$) gives

$$\underbrace{\frac{Q}{4}S(x_s)(1-\alpha_s)}_{\text{absorbed shortwave}} - \underbrace{C(T_s - \overline{T})}_{\text{flux divergence}} = A + BT_s \tag{7}$$

where $\alpha_s$ is the albedo exactly at the ice-line. A simple choice is to take $\alpha_s = \frac{1}{2}(\alpha_1 + \alpha_2)$ (e.g., Lindzen, 1990). From equation (7), and by construction of the model, it is seen that the OLR at the ice-line is always a constant. The combination of the other two terms in the energy balance – the absorbed shortwave radiation minus the divergence of the poleward heat flux – must equal this constant.

Equation (7) can be combined with (2) to eliminate $\overline{T}$:

$$\frac{Q}{4}(1-\alpha_s)S(x_s) + \frac{Q}{4}\frac{C}{B}(1-\alpha_p) = \text{constant}. \tag{8}$$

Substituting from (5) into (8) gives an analytical expression for $Q(x_s)$ (e.g., Lindzen, 1990) that governs how the equilibrium ice-line varies as a function of solar constant, $Q$ (Figure 2a). Figures like 2a appear in many papers on Snowball Earth. Some of these studies argue on physical grounds and others provide detailed (and sometimes involved) mathematical proofs that no stable solution is possible when the slope of $x_s$ vs. $Q$ is negative (e.g., Held and Suarez, 1974; North, 1975; Ghil, 1976; Su and Hsieh, 1976; Drazin and Griffel, 1977; Lindzen and Farrell, 1977; Cahalan and North, 1979; North, 1990; Shen and North, 1999). The term 'slope-stability theorem' has been coined to describe the proposition.

We show in the next section that a formal analysis of the ice-albedo feedback provides a simple poof of the slope-stability theorem, and gives physical insight into the cause of the instability.

#### 2.1.2 Feedback analysis from the ice-line perspective

The instability results from the variation of albedo with changing climate state (as represented by $x_s, \overline{T}$). One way to evaluate the effect of this is to ask: what is the difference between the sensitivity of the ice-line latitude to variations in the solar constant with and without albedo variations? Framing the issue in this way is at the heart of a feedback analysis (e.g., Roe, 2009).

A first-order Taylor series expansion of (8) gives:

$$\Delta Q \left\{ \frac{(1-\alpha_s)S(x_s)}{4} + \frac{1}{4}\frac{C}{B}(1-\alpha_p) \right\} + \Delta x_s \left\{ \frac{Q(1-\alpha_s)}{4} S'(x_s) \right\} - \Delta x_s \frac{QC}{4B}\alpha_p' = 0, \qquad (9)$$

where the primes denote derivatives with respect to $x_s$. First, consider the case in which no albedo variations are permitted. In this instance $\alpha_p' = 0$ and the sensitivity of the ice-line to insolation can be written as:

$$\Delta x_s = \lambda_x \Delta Q, \tag{10}$$

where

$$\lambda_x = -\frac{(1-\alpha_s)S(x_s) + \frac{C}{B}(1-\alpha_p)}{Q(1-\alpha_s)S'(x_s)}. \tag{11}$$

$S'$ is negative and so $\lambda_x$ is positive. $\lambda_x$ can be straightforwardly calculated from previous expressions.

Secondly, consider the case in which albedo variations are permitted. Now $\alpha_p' \neq 0$ in equation (9), and variations in $x_s$ can be written as

$$\Delta x_s = \frac{\lambda_x}{1-f_x}\Delta Q, \tag{12}$$

where

$$f_x = \frac{C\alpha_p'}{BS'(1-\alpha_s)}. \tag{13}$$

$f_x$ is the feedback factor in this problem (e.g., Roe, 2009). Both $\alpha_p'$ and $S'$ are negative so, as expected, $f_x$ is a positive feedback.

Catastrophe occurs in the limit $f \rightarrow 1$. Equation (13) demonstrates that, provided there is some poleward heat transport (i.e., $C \neq 0$), this instability must be present for all parameter values: since $S'$ tends to zero as $x_s$ nears the equator (Fig. 1a), at some latitude $f$ must exceed one. The slope stability theorem also follows directly from (12): for $f_x < 1$ (i.e., stable equilibria), $\Delta x_s / \Delta Q > 0$; for $f_x > 1$ (i.e., unstable equilibria), $\Delta x_s / \Delta Q < 0$. This behavior is shown in Figure 2b.

#### 2.1.3 What is the physical explanation of the instability?

The mechanism of the instability can be understood physically as follows. Suppose, beginning from some equilibrium climate state, the ice-line advances while $Q$ is held constant. The higher local insolation at lower latitudes produces warming at the perturbed ice-line position. Acting alone, this warming would tend to restore the ice-line to its previous equilibrium position. However, the local divergence of heat flux increases at lower latitudes, and this produces cooling at the new ice-line position. If the cooling is larger than the warming, the ice-line will continue to advance, and hence the situation is unstable. We can see this from the following: the relative magnitude of these two tendencies can be found by differentiating the terms in equation (7) with respect to $x_s$ and holding $Q$ constant. The ratio $R$ of the cooling tendency (i.e., the increase of local heat flux divergence) to the warming tendency (i.e., the increase in local insolation) can then be written as:

$$R = \frac{C \left. \frac{d\overline{T}}{dx_s} \right|_Q}{\frac{Q(1-\alpha_s)}{4} \frac{dS}{dx_s}}. \tag{14}$$

From equation (2), $d\overline{T}/dx_s|_Q = -\frac{Q}{4B}(d\alpha/dx_s)$, and so $R$ becomes

$$R = \frac{-C\alpha_p'}{B(1-\alpha_s)S'} \equiv f_x. \tag{15}$$

Therefore, for an incremental advance of the ice-line, the cooling term exceeds the warming term at the same latitude that $f_x$ exceeds 1. Thus we also see that the local and global perspectives on the feedback are equivalent.

The snowball instability is inevitable in this climate model simply because of the geometry of a sphere. The rate at which the local insolation increases (or in other words, the restoring warming tendency described above) diminishes as the ice-line latitude moves equatorwards (i.e., Figure 1a), while the destabilizing effect of the local divergence of heat flux increases. As the equilibrium ice-line descends to lower and lower latitudes it becomes easier and easier for a perturbed ice-line to advance. Thus the strengthening of this positive albedo feedback as the ice line advances reflects a robust property of the climate system, and so is likely to hold in more sophisticated models. We note that Lindzen and Farrell (1977, 1980), Poulsen et al. (2001) and others have explored how including dynamical circulation regimes such as the Hadley Cell or additional heat-transport processes, such as ocean circulation, can modify this picture and we broach this further in the discussion.

#### 2.1.4 What is the dependency of the instability on physical parameters?

Differentiating equation (5) with respect to $x_s$, and substituting into equation (13) gives

$$f_x = -\frac{C\alpha_p'}{B(1-\alpha_s)S'} = -\frac{C(\alpha_1-\alpha_2)S(x_s)}{B(1-\alpha_s)S'}. \tag{16}$$

The strength of the feedback therefore depends linearly on the albedo contrast between ice-covered and ice-free areas, as is perhaps intuitive. $f_x$ is also proportional to $C$ – the more efficiently heat is redistributed, the stronger the feedback. In effect, this reflects that heat can be 'pulled out' of the tropics more effectively, thereby creating a greater cooling tendency and permitting the ice-line to advance more easily (see also Held and Suarez, 1974). This has a strong physical basis, and so it is likely to also be true of models that have a more sophisticated representation of heat transport. Finally, $f_x$ is inversely proportional to $B$, since as noted above, a higher value of $B$ means a lower sensitivity of climate to perturbations. We note that all of the model parameters enter into $f$ at the same order, implying they have equal importance.

Setting $f_x = 1$ in equation (16) produces a quadratic equation for the sine of the latitude, $x^*$, at which the instability occurs:

$$\frac{3s_2}{2}x^{*2} + 2s_2\frac{C(\alpha_1-\alpha_2)}{B(1-\alpha_s)}x^* + (1+\frac{s_2}{2}) = 0. \tag{17}$$

The quadratic nature of the equation and the presence of $s_2$ (the coefficient in the series expansion of the insolation distribution) reflect the spherical geometry. Note the model parameters appear only as a factor in the linear term in equation 17, and in the same nondimensional combination as in equation (16). An increase in this linear factor causes an increase in $x^*$ (i.e., the instability occurs at a higher latitude), reflecting a less stable system. Following the arguments of the previous section, the latitude of the instability is also the latitude of the ice line at which the net incoming energy fluxes are independent of $x_s$: equatorwards of this latitude, an advance of the ice line leads to a net cooling at the ice-line; polewards of this latitude, an advance of the ice line leads to a net warming at the ice-line.

Does this work to sate Reviewer C?

#### 2.1.5 Feedback analysis from the global temperature perspective

The magnitude of a feedback within a system can depend on the variable or field of interest (e.g., Roe, 2009). This can be illustrated by recasting the EBM system to solve for global-mean temperature instead of ice-line latitude. This makes the problem closer to the normal definition of the climate sensitivity to a radiative perturbation (e.g., Charney et al., 1979; Knutti and Hegerl, 2008; Roe, 2009).

Now we solve for changes in $\overline{T}$ due to changes in $Q$. First, suppose again that there is no albedo feedback (i.e., $\alpha_p' = 0$). In this case, from (2), first-order perturbations in temperature and solar constant are related by

$$\Delta\overline{T} = \lambda_T \Delta Q, \tag{18}$$

where

$$\lambda_T = \frac{(1-\alpha_p)}{4B}. \tag{19}$$

This is the equivalent of the standard climate sensitivity parameter for this problem (e.g., Roe, 2009), though in this case it is the sensitivity to changes in solar constant, not to imposed independent forcing due to $CO_2$. $\lambda_T^{-1}$ measures the basic stabilizing tendency in the energy balance of the planet, whereby the outgoing longwave radiation acts to restore temperatures back to equilibrium after a perturbation. For a given change in insolation, a higher value of $\lambda_T^{-1}$ means a smaller temperature change, and so reflects a stronger damping tendency. Defined in this way, climate sensitivity decreases in a colder climate because as the planetary albedo increases, a given increment in insolation produces less radiative forcing in terms of what is actually absorbed.

Now if instead the albedo is allowed to vary with temperature, the right-hand side of the equation must include the additional radiative perturbation that occurs in response to the change in albedo:

$$\begin{aligned} \Delta\overline{T} &= \lambda_T \Delta Q - \frac{Q}{4B}\frac{d\alpha_p}{d\overline{T}}\Delta\overline{T} && (20)\\ &= \lambda_T \Delta Q - \frac{Q}{4B}\alpha_p'\frac{\Delta x_s}{\Delta\overline{T}}\Delta\overline{T}. && (21) \end{aligned}$$

This last term on the right hand side is the albedo feedback. Solving for $\Delta\overline{T}$ explicitly gives

$$\Delta\overline{T} = \frac{\lambda_T}{1-f_T}\Delta Q, \tag{22}$$

where $f_T$ is the albedo feedback factor (e.g., Roe, 2009), and is given by

$$f_T = -\frac{Q}{4B}\alpha_p' \frac{\Delta x_s}{\Delta \overline{T}} = -\frac{\frac{Q}{4}\alpha_p'}{B\frac{\Delta \overline{T}}{\Delta x_s}}, \tag{23}$$

where the $\Delta$ notation means that the derivative is calculated along the curve $x_s = x_s(Q, \alpha_p'))$ calculated from eq (12) .

As with any positive feedback, (23) reflects competing tendencies on a conservation equation (e.g., Roe, 2009). In this case, the numerator on the right hand side reflects the destabilizing process of the albedo increasing as the ice-line advances equatorwards, and the denominator reflects the stabilizing process of changes in the longwave radiation to space. Equation (23) is quite general and could readily be diagnosed from perturbation experiments using global climate models, for example. The relationship between the ice-line feedback and the global temperature feedback comes from the following:

$$\frac{\Delta \overline{T}}{\Delta Q} = \frac{\partial \overline{T}}{\partial \alpha_p}\alpha_p' \frac{\Delta x_s}{\Delta Q} + \left.\frac{\partial \overline{T}}{\partial Q}\right|_{\alpha_p = const}. \tag{24}$$

which can be rewritten as:

$$\frac{\lambda_T}{1-f_T} = -\frac{Q\alpha_p'}{4B}\left(\frac{\lambda_x}{1-f_x}\right) + \lambda_T. \tag{25}$$

From this equation it is straightforward to demonstrate that $f_x$ and $f_T$ both cross 1 at the same ice-line latitude, shown in Figure 2b.

## 2.2 Diffusive energy balance models

North (1975) suggested an alternative, and arguably somewhat more physical, parameterization for the poleward heat flux, proposing that it be parameterized as proportional to the local meridional temperature gradient. In this case $\nabla \cdot \vec{F}$ in (1) is given by

$$\nabla \cdot \vec{F} = -D \frac{d}{dx}(1 - x^2)\frac{dT}{dx}. \tag{26}$$

### 2.2.1 Traditional Analysis

North (1975) demonstrated that an accurate analytical approximation to equations (1) and (26) could be obtained using hypergeometrical functions and matching boundary conditions at the ice-line. North (1975), Cahalan and North (1979), and Shen and North (1999) and others have studied the stability properties of these solutions, analyzing the time-dependent behavior of perturbations away from the derived equilibrium solutions.

Figure 3a reproduces the original analytical solutions derived by North (1975) using his chosen parameter set (which are slightly different from those used up to this point in this paper). From the slope of $x_s$ vs. $Q$ it is clear that stable climates do not exist equatorwards of $x_s \approx 0.6$. In addition, there is also a striking phenomenon polewards of $x_s \approx 0.95$, the so-called 'small ice cap instability' (e.g., North 1984): beyond some latitude, the slope of $x_s$ vs. $Q$ turns negative, implying that the polar ice-cap can only be stable if it extends past some finite latitude. The reasons for this behavior has been analyzed in detail in simple systems (e.g., Lindzen and Farrell, 1977; North 1984), though its presence in more complete climate models is still discussed (e.g., Crowley et al., 1994; Lee and North, 1995; Langen and Alexeev, 2004; Rose and Marshall, 2009; Enderton and Marshall, 2009).

### 2.2.2 Feedback Analysis

A simple alternative to these time-dependent analyses is to calculate the feedback strengths by direct substitution of the analytical solutions provided in North (1975) into equations (18), (23), and (25). Figure 3d shows both $f_x$ and $f_T$. $f_x$ behaves as expected - it lies between zero and one in the stable ice-line regime, and exceeds one for unstable ice-line regimes. The behavior of $f_T$ is more interesting. It goes through two singularities, and actually becomes negative near the equator and near the pole.

The cause of this peculiar behavior is related to the small ice cap instability and, as it turns out, there is a directly analogous counterpart near the equator. The explanation closely follows arguments in Lindzen and Farrell (1977) for the small ice-cap instability, and is illustrated schematically in Figure 3. Three curves are shown for equilibrium climate states using the Budyko-style approximation for $\nabla \cdot \vec{F}$, but using different values for the ice-line albedo ((i) $\alpha_s = \alpha_1$; (ii) $\alpha_s = 0.5*(\alpha_1+\alpha_2)$ as has been used up to now; (iii) and $\alpha_s = \alpha_2$).

The small ice-cap instability can be understood by considering the intersection of these curves with $x_s = 1$. Recall that these curves give pairs of $(x_s, Q)$ that are equilibrium solutions of the model equations, and that the stability of these equilibrium states can be judged from whether $dx_s/dQ > 0$ (stable) or $dx_s/dQ < 0$ (unstable). Imagine starting with an ice-free Earth and high $Q$ (point $A_1$ in Figure 4). If $Q$ is now gradually lowered, the system moves toward point $A_2$. As soon as any ice forms on the planet, though, the solution trajectory must jump from $\mathrm{A}_2$ to $\mathrm{A}_3$, because of the discontinuity in albedo. The introduction of any ice at all means, somewhat counterintuitively, that the solar constant must be increased to maintain the ice at that latitude in equilibrium. As pointed out by Lindzen and Farrell (1977), in the Budyko-style EBM the non-local nature of the heat transport means the discontinuity is confined to $x_s = 1$. For North-style diffusion however, the influence of the albedo discontinuity leads to a boundary layer that extends into the domain with a characteristic length scale equal to $\sqrt{D/B}$ (see also North, 1984). This is illustrated schematically by the thick curve. Along this trajectory of equilibrium, albeit unstable, climates from $\mathrm{A}_2$ to $\mathrm{A}_3$, $Q$ and $T$ are both increasing (Figure 2b), even though the ice line is descending equatorwards. Thus the gradient $\Delta\overline{T}/\Delta x_s$ is negative (Figure 2c) and so from (23), $f_T$ is also negative.

There is a directly analogous discontinuity at the equator. Start with an ice-covered Earth and low $Q$ (point $B_1$). If $Q$ is now gradually increased then the system moves along the path $\mathrm{B}_1$ to $\mathrm{B}_2$ . But again, as soon as any ice-fee areas emerge the solution trajectory must jump to $\mathrm{B}_3$. Following the same reasoning as before, $\Delta\overline{T}/\Delta x_s$ reverses (Figure 3c), and so $f_T$ is negative. The thick green line in Figure 3c also indicates schematically the penetration of the impact of this discontinuity into the domain for North-style diffusive transport. The equatorial discontinuity is not readily apparent in the $x_s$ vs. $Q$ curves because the slope of the curve from $B_2$ to $B_3$ has the same sense as the slope of $dx_s/dQ$ at slightly higher latitudes. Taken together, the polar and the near-equator instabilities produce the thick green curve in Figure 4, which is similar to the curve of $x_s$ vs. $Q$ curve in Figure 3a.

In summary, imagine a global temperature increase from an unspecified cause. For most values of $x_s$, this causes a retreat of the ice-line amplifying the original warming (Figure 3c and Equation 20). However, in the vicinity of the equator and pole, the discontinuity in albedo exerts a stronger control on the system dynamics, and the warming is in fact associated with an advance of the ice line. This damps the original warming and so the feedback is negative. Although this only occurs here in equilibrium climate states that are unstable, it is an exotic illustration of the point that if the dominant physical processes change as a function of mean climate state, the magnitude and even the sign of the feedback can vary (e.g., Roe, 2009).

# 3 Discussion

In essence, the analysis presented here recasts existing solutions for simple energy balance models into the language of feedback analysis. In doing so, the physical cause of the Snowball Earth instability can be clearly and simply laid out. From the perspective of the global energy balance, the strength of the feedback is determined by the competition between the stabilizing tendency of the outgoing longwave radiation, and the destabilizing tendency of less radiation being absorbed as the planet brightens. From the perspective of the ice-line, the feedback is the ratio of changes in local insolation and in the divergence of the poleward heat flux as $x_s$ changes.

Our analysis enables derivation of simple expressions for the strength of the albedo feedback as a function of mean climate state and choice of climate parameters. One principal control is of course the spherical geometry of the Earth which, at least within the strictures of these simple models, makes the instability inevitable at some latitude. In the case of the Budyko-style model the latitude of the ice-line instability also depends on a simple non-dimensional combination of model parameters.

We have investigated the apparently strange result that, for diffusive parameterizations of heat flux, the ice albedo can even act as a negative feedback (i.e., have a stabilizing effect) on global temperature variations. It happens here only for climate states that are unstable, because of the very tight coupling assumed between the ice-line and temperature in the energy balance model. However the result that global mean temperature might have a minimum at a nonzero ice-line latitude because of the albedo discontinuity is quite physical. It remains to be explored whether this negative ice-albedo feedback is just a curiosity of these particular models, or if it can help explain the occurrence of equilibrium 'slush-ball' states (i.e., an ice-free equatorial band) found in some climate models (e.g., Hyde et al., 2000; Crowley et al., 2001), and which has been argued to be more consistent with geological evidence (Allen and Etienne, 2008). Another useful diagnostic is suggested by the results in Sections 2.1.4 and 2.1.3. When the overall climate is stable it is because an equatorward advance of the ice-line causes a net warming at the ice-line. This is likely a very general result. Studying the energy budget response to an ice-line perturbation in models that exhibit slush-ball states would elucidate which terms are responsible for that warming, and perhaps therefore explain the differences from models that do not exhibit slush-ball states.

The very concept of a feedback implicitly partitions the system into a reference state, and a set of physical 'feedback' processes (e.g., Roe, 2009). In this context, having an ice-albedo feedback means introducing a process that allows the albedo to vary with climate state. A straightforward lesson that also applies to more complex systems is that the impact of adding this process depends on which part of the system is of interest. In this simple case studied here, the feedback strength is different for the global-mean temperature and for the ice-line.

Our representations of the feedbacks by ratios of derivatives illustrates the general feature of feedbacks; whereas in the simplified physical system considered in this paper the derivatives were taken with respect to the spatial variable $x_s$, the primes could more generally indicate derivatives taken with respect to other climate variables such as circulation pattern, atmospheric composition, etc.

The zonal-mean EBMs presented here are obviously highly idealized representations of the real world. Severe approximations have been made in their derivation, not the least of which are the absence of clouds and a seasonal cycle, and these approximations render the albedo feedback as being substantially larger than is inferred from GCMs for the modern climate (e.g., Soden and Held, 2006). It would be of interest to diagnose ice-albedo feedbacks within GCMs as the solar constant is reduced (following, for example, the methods of Soden and Held, 2006), and to evaluate if the feedback strength varies in ways that are consistent with the predictions from (16). It may well be that the reason the solar constant must be lowered in GCMs by significantly more than would be suggested by the EBMs (e.g., Poulsen and Jacob, 2004; Voigt and Marotzke, 2009) is due to the weaker ice-albedo feedback in the GCMs. The consequence of a weaker albedo feedback are predicted in Equations (16) and (17).

Some studies have suggested that there might be a 'stability ledge' due to the effects of the Hadley cell (Lindzen and Farrell, 1977; some climate model results suggests that ocean transports (Poulsen et al., 2001) or latent heat fluxes (Poulsen, 2003) can act to inhibit a complete glaciation. Poulsen and Jacob (2004) analyze such effects in some detail. These processes could in principal be cast as additional feedbacks in the energy budget. To first order, the net effect on the climate is given by the sum of the individual feedback factors and so isolating just the ice-albedo feedback provides a guide for how strong those negative feedbacks have to be to create a stable equilibrium (i.e., the sum must be less than one).

Recent advances in feedback analysis permit the full spatial structure of climate feedbacks to be calculated (e.g., Soden et al., 2008), and can even include ocean heat uptake (Gregory and Forster, 2008). A full feedback diagnosis of the simulations from more complicated models such as Voigt and Marotzke (2009) would permit the relative importance of individual processes in these models (and the uncertainties in them) to be propagated through the system dynamics. One important and robust expectation is that uncertainties in physical process (and in model parameterizations of them) lead to large uncertainties in the system response in the vicinity of $f = 1$, because of the strong amplification that is occurring (e.g., Roe, 2009). It is perhaps not surprising then, that GCMs exhibit such a diversity of behavior (e.g., Voigt and Marotzke, 2009).

The Snowball Earth phenomenon illustrates how localized physical processes can have a global impact. Here, strong model assumptions control how something happening at one particular latitude (the albedo changing because of an ice-line advance) acts to affect the global-mean climate. In nature other important feedbacks are also localized, such as the strong negative feedback of subtropical stratus decks (e.g., Sanderson et al., 2008), or the high-latitude deep ocean heat uptake (e.g., Gregory and Forster, 2008; Winton et al., 2009; Baker and Roe, 2009). Perhaps one important way forward for improving both global and regional climate predictions will be to better understand how these regional processes combine to give the full, global, system response.

# Acknowledgements

We are grateful to Steve Warren for helpful comments and suggestions, to Gerald North and two other constructive reviewers, and to Anthony Broccoli, the editor.

# References


Abramowitz, M., and I. A. Stegun, 1965: Handbook of mathematical functions with formulas,graphs, and mathematical tables. Dover publications, New York, pp 1046.

Allen, P.A., and J.L. Etienne, 2008: A sedimentary challenge to Snowball Earth. *Nature Geo.*, **1**, 817-825.

Baker M.B., and G.H. Roe, 2009: The shape of things to come: why is climate change so predictable? *J. Climate*, **22**, 4574-4589.

Baum, Steven K.; Crowley, T.J, 2001: GCM response to Late Precambrian (590 Ma) icecovered continents. *Geophys. Res. Lett.*, **28**, 583-586.

Budyko, M.I., 1969: The effect of solar radiation variations on the climate of the Earth, *Tellus*, **5**, 611-619.

Cahalan, R.F., and G.R. North, 1979: A stability theorem for energy-balance climate models. *J. Atmos. Sci.*, **36**, 1178-1188.

Chylek, P., and J.A. Coakley, 1975: Analytical analysis of a Budyko-type climate model. *J. Atmos. Sci*,**32**, 675-679.

Coakley, J. A., 1979: A study of climate sensitivity using a simple energy balance model. *J. Atmos. Sci.*, **36**, 260-269.

Crowley, T. J., and S. K. Baum, 1993: Effect of decreased solar luminosity on late Precambrian ice extent, *J. Geophys. Res.*, **98**, 16,723-16,732.

Crowley, T.J., K.-J. Yip, and S.K. Baurn, 1994: Snowline instability in a general circulation model: Application to Carboniferous glaciation. *Clim. Dyn.*, **10**, 363-376.

Crowley, T.J., W.T. Hyde, and W.R. Peltier, 2001: $CO_2$ levels required for deglaciation of a Near-Snowball Earth. *Geophys. Res. Lett.*, **28**, 283286.

Drazin, P. G., and D. H. Griffel, 1977: On the branching structure of diffusive climatological models. *J. Atmos. Sci.*, **34**, 1858-1867.

Enderton, D. and J. Marshall, 2009: Controls on the total dynamical heat transport of the atmosphere and oceans: to appear *J. Atmos.Sci.*

Fairchild, I.J., and M.J. Kennedy, 2007: Neoproterozoic glaciation in the Earth System, *J. Geol. Soc., London*, **164**, 895921.

Ghil, M., 1976: Climate stability for a Sellers-type model. *J. Atmos. Sci.*, **33**, 3-20.

Gregory J.M.,P.M. Forster, 2008: Transient climate response estimated from radiative forcing and observed temperature change. *J. Geophys. Res*, **113**: D23105.

Held, I. M., and M. Suarez, 1974: Simple albedo feedback models of the icecaps. *Telus*, **36**, 613-629.

Hoffman, P.F., A.J.Kaufman, G.P Halverson, and D.P. Schrag, 1998: A Neoproterozoic snowball Earth. *Science*, **281**, 1342-1346.

Hoffman, P.F., and D.P Schrag, 2002. The Snowball Earth hypothesis: Testing the limits of global change. *Terra Nova*, **14**, 129-155.

Hoffman, P.F., and Z.-X. Li, 2008: A palaeogeographic context for Neoproterozoic glaciation. *Palaeogeography, Palaeoclimatology, Palaeoecology*, **277**, 158-172.

Hyde, W.T., T.J. Crowley, S.K. Baum, and W.R. Peltier, 2000. Neoproterozoic snowball Earth simulations with a coupled climate/ice-sheet model. *Nature*, **405**, 425-429.

Jenkins, G.S., and Frakes, L.A., 1998. GCM sensitivity test using increased rotation rate, reduced solar forcing and orography to examine low latitude glaciation in the Neoproterozoic. *Geophys. Res. Lett.*.**25**, 3525-3528.

Jenkins, G.S. and S.R. Smith, 1999: GCM simulations of Snowball Earth conditions during the late Proterozoic. *Geophys. Res. Lett.* **26**, 2263-2266.

Kerr, R.A., 2000: An Appealing Snowball Earth That's Still Hard to Swallow. *Science*, **287**, 1734 - 1736.

Kirschvink, J. L, 1992: in *The Proterozoic Biosphere.* eds Schopf, J. W. and Klein, C. Cambridge Univ. Press, 51-52.

Lee, W.H. and G. R. North, 1995: Small ice cap instability in the presence of fluctuations. Climate Dynamics, 11, 242-246.

Lindzen, R.S., and B. Farrell, 1977: Some realistic modifications of simple climate models. *J. Atmos. Sci.*, **34**, 1487-1501.

Lindzen, R.S., and B. Farrell, 1980: The role of polar regions in global climate, and the parameterization of global heat transport. *Mon. Wea. Rev.*, **108**, 2064-2079.

Lindzen, R.S., 1990: Dynamics in Atmospheric Physics, Cambridge University Press, Cambridge, UK, 320pp.

North, G. R, 1975: Theory of energy-balance climate models. *J. Atmos. Sci.*, **32**, 2033-2043.

North, G. R., R. F. Cahalan, and J. A. Coakley, 1981: Energy balance climate models. *Rev. Geophys. Space Phys.*, **19**, 91121.

North, G.R., 1990: Multiple solutions in energy balance climate models. *Global and Planetary Change*, **2**, 225-235.

Poulsen, C.J., R.T. Pierrehumbert, and R.L. Jacob, 2001: Impact of ocean dynamics on the simulation of the Neoproterozoic snowball Earth *Geophys. Res. Lett.*, **28**, 1575-1578.

Poulsen, C.J., 2003: Absence of a runaway ice-albedo feedback in the Neoproterozoic, *Geology*, **31**, 473-476.

Poulsen C.J., and R.L. Jacob, 2004: Factors that inhibit snowball Earth simulation. *Paleoceanography*, **19**(4):PA4021

Roe, G.H., 2009: Feedbacks, timescales, and seeing red. *Annual Reviews of Earth Plan. Sci.*, **37**, 93-115.

Rose, B. and J. Marshall, 2009: Ocean heat transport, sea-ice and multiple climate states: insights from energy balance models. *J. Atmos. Sci.*, 10.1175/2009JAS3039.1

Sanderson, B. M., C. Piani, W. Ingram, D. A. Stone, and M. R. Allen, 2008b: Towards constraining climate sensitivity by linear analysis of feedback patterns in thousands of perturbed-physics GCM simulations. *Clim Dyn*, **30(2-3)**, 175190.

Sellers, W.D., 1969: A climate model based on the energy balance of the earth-atmosphere system. *J. Appl. Meteorol.*, **8**, 392-400.

Shen, S., and G.R. North, 1999: A simple proof of the slope stability theorem for energy balance climate models. *Canadian Apl. Math. Quaterly*, **7**, 203-215.

Soden, B.J. and I.M. Held, 2006: An assessment of climate feedbacks in coupled oceanatmosphere models. *J. Climate*, **19**, 33543360.

Soden, B.J., I.M. Held, R. Colman, K.M. Shell, J.T. Kiehl, and C.A. Shields, 2008: Quantifying climate feedbacks using radiative kernels. J. Climate, **21**, 3504-3520.

Su, C.H., and D.Y. Hsieh, 1976: Stability of the Budyko Climate Model. *J. Atmos. Sci.*, **33**, 2273-2275.

Voigt, A. and J. Marotzke, 2009: The transition from the present-day climate to a modern Snowball Earth. Climate Dynamics, DOI 10.1007/s00382-009-0633-5

Winton, M., K. Takahashi, and I. Held, 2009: Importance of ocean heat uptake efficacy to transient climate change. *Submitted to Journal of Climate.*

# List of Figures

**Figure 1** a) Normalized insolation distribution $S(x)$ as a function of latitude. The normalization is such that $\int_0^1 S(x)dx = 1$; b) planetary albedo, $\alpha_p(x_s)$ as a function of the latitude of the ice-line. Note that the x-axes in the two panels refer to different things.

**Figure 2** Properties relating to the ice-line instability in the Budyko model. a) Equilibrium ice-line as a function of insolation relative to modern. Following Lindzen and Farrell (1977), $Q_0 = 1336\ \mathrm{W\ m^{-2}}$: regions with positive slope are stable equilibria, negative slopes are unstable equilibria; b) albedo feedback factors $f_x$, $f_T$. Only regions with $f < 1$ are stable equilibria.

**Figure 3** Properties of solutions to the North diffusive EBM. a) the ice-line as a function of $Q/Q_0$, using North (1975) analytical solutions and parameters. The thin lines show turning points; b) global mean temperature vs. solar constant for the same solution; c) global mean temperature vs. ice-line for the same solution; d) $f_x$ and $f_T$. The thin lines confirm that the feedbacks exceed 1 at the latitude of the turning points in panel a). Note that $f_T$ becomes negative near the equator and the pole. See text for the explanation.

**Figure 4** Schematic explanation of small ice-cap instability, and the regions of negative $f_T$ feedback, extending the arguments of Lindzen and Farrell (1977). See text for details.

# 4 Figures

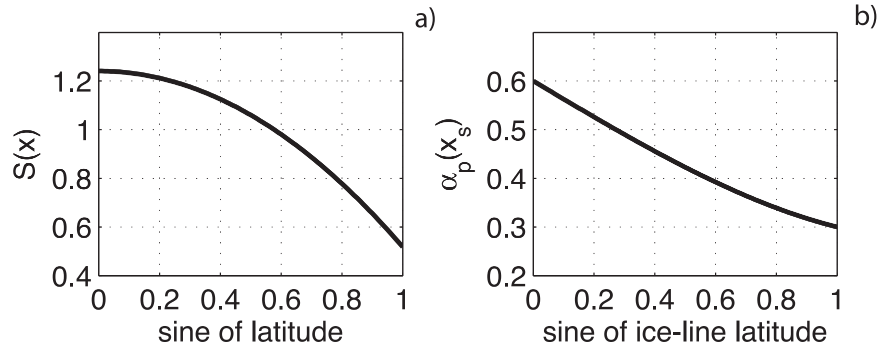


Figure 1: a) Normalized insolation distribution $S(x)$ as a function of latitude. The normalization is such that $\int_0^1 S(x)dx = 1$; b) planetary albedo, $\alpha_p(x_s)$ as a function of the latitude of the ice-line. Note that the x-axes in the two panels refer to different things.

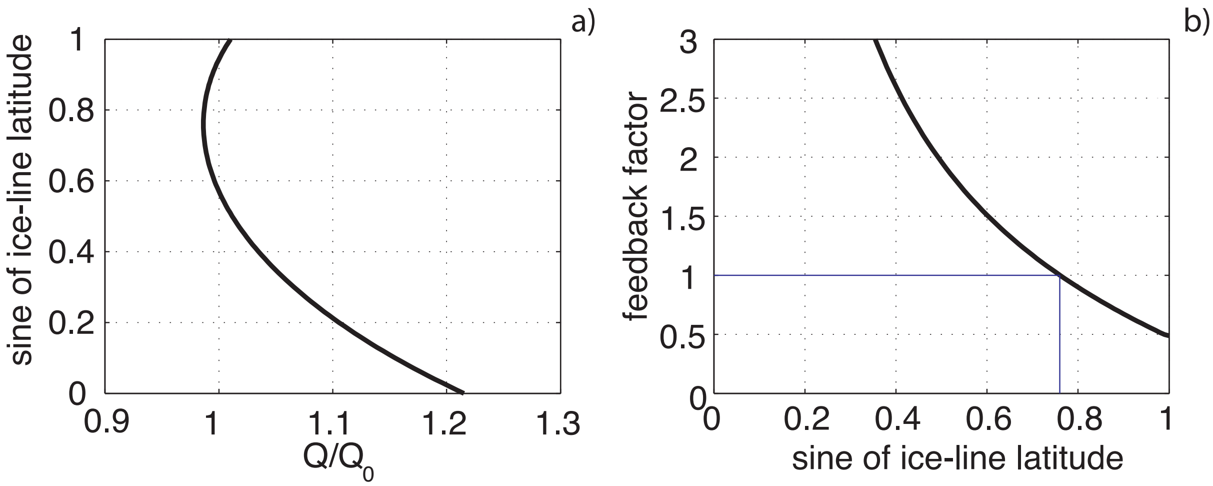


Figure 2: Properties relating to the ice-line instability in the Budyko model. a) Equilibrium ice-line as a function of insolation relative to modern. Following Lindzen and Farrell (1977), $Q_0 = 1336$ W m$^{-2}$: regions with positive slope are stable equilibria, negative slopes are unstable equilibria; b) albedo feedback factors $f_x$, $f_T$. Only regions with $f < 1$ are stable equilibria.

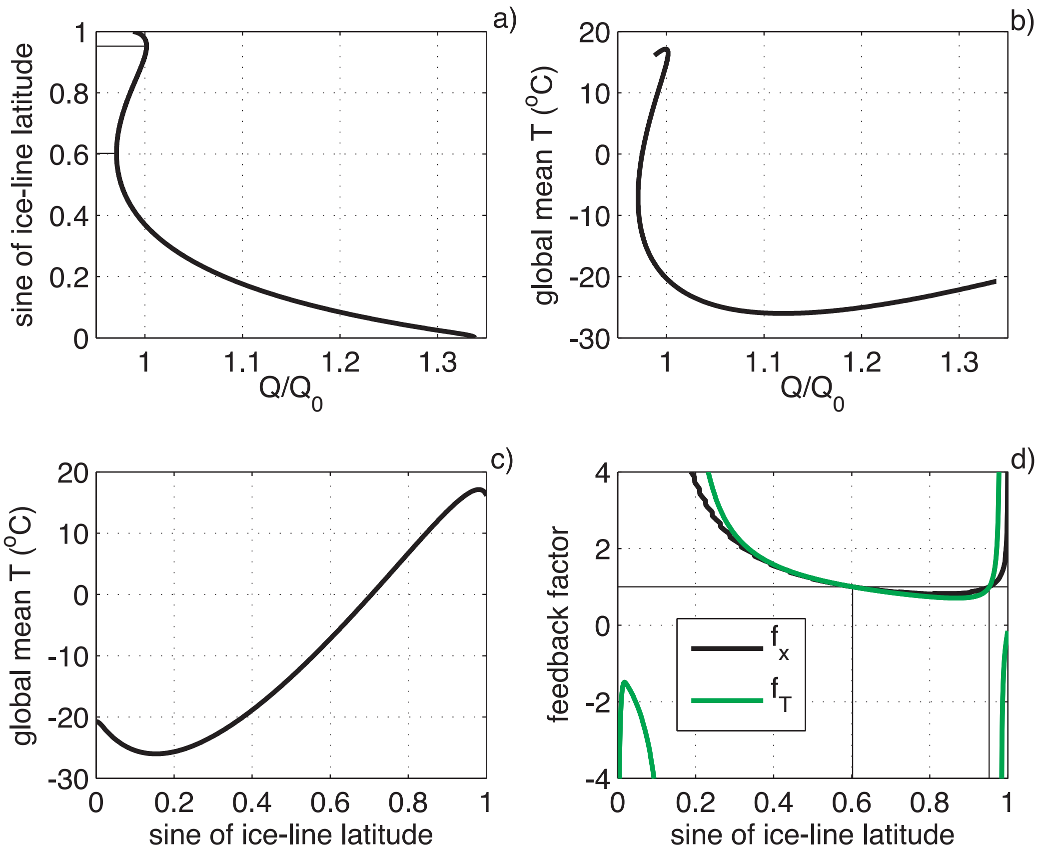


Figure 3: Properties of solutions to the North diffusive EBM. a) the ice-line as a function of $Q/Q_0$, using North (1975) analytical solutions and parameters. The thin lines show turning points; b) global mean temperature vs. solar constant for the same solution; c) global mean temperature vs. ice-line for the same solution; d) $f_x$ and $f_T$. The thin lines confirm that the feedbacks exceed 1 at the latitude of the turning points in panel a). Note that $f_T$ becomes negative near the equator and the pole. See text for the explanation.

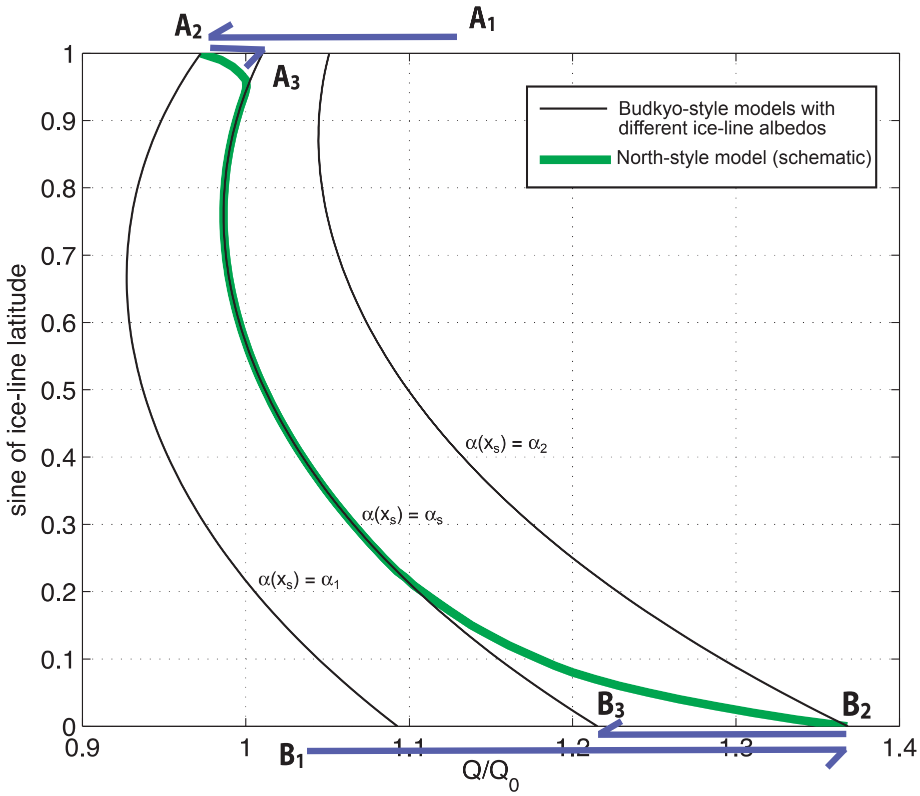


Figure 4: Schematic explanation of small ice-cap instability, and the regions of negative $f_T$ feedback, extending the arguments of Lindzen and Farrell (1977). See text for details.