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From the Preface to the First Printing

A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery. Such experiences at a susceptible age may create a taste for mental work and leave their imprint on mind and character for a lifetime.

Thus, a teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking.

Also a student whose college curriculum includes some mathematics has a singular opportunity. This opportunity is lost, of course, if he regards mathematics as a subject in which he has to earn so and so much credit and which he should forget after the final examination as quickly as possible. The opportunity may be lost even if the student has some natural talent for mathematics because he, as everybody else, must discover his talents and tastes; he cannot know that he likes raspberry pie if he has never tasted raspberry pie. He may manage to find out, however, that a mathematics problem may be as much fun as a crossword puzzle, or that vigorous mental work may be an exercise as desirable as a fast game of tennis. Having tasted the pleasure in mathematics he will not forget it easily and then there is a good chance that mathematics will become something for him: a hobby, or a tool of his profession, or his profession, or a great ambition.

The author remembers the time when he was a student himself, a somewhat ambitious student, eager to understand a little mathematics and physics. He listened to lectures, read books, tried to take in the solutions and facts presented, but there was a question that disturbed him again and again: "Yes, the solution seems to work, it appears to be correct; but how is it possible to invent such a solution? Yes, this experiment seems to work, this appears to be a fact; but how can people discover such facts? And how could I invent or discover such things by myself?" Today the author is teaching mathematics in a university; he thinks or hopes that some of his more eager students ask similar questions and he tries to satisfy their curiosity. Trying to understand not only the solution of this or that problem but also the motives and procedures of the solution, and trying to explain these motives and procedures to others, he was finally led to write the present book. He hopes that it will be useful to teachers who wish to develop their students' ability to solve problems, and to students who are keen on developing their own abilities.

Although the present book pays special attention to the requirements of students and teachers of mathematics, it should interest anybody concerned with the ways and means of invention and discovery. Such interest may be more widespread than one would assume without reflection. The space devoted by popular newspapers and magazines to crossword puzzles and other riddles seems to show that people spend some time in solving unpractical problems. Behind the desire to solve this or that problem that confers no material advantage, there may be a deeper curiosity, a desire to understand the ways and means, the motives and procedures, of solution.

The following pages are written somewhat concisely, but as simply as possible, and are based on a long and serious study of methods of solution. This sort of study, called *heuristic* by some writers, is not in fashion nowadays but has a long past and, perhaps, some future.

Studying the methods of solving problems, we perceive another face of mathematics. Yes, mathematics has two faces; it is the rigorous science of Euclid but it is also something else. Mathematics presented in the Euclidean way appears as a systematic, deductive science; but mathematics in the making appears as an experimental, inductive science. Both aspects are as old as the science of mathematics itself. But the second aspect is new in one respect; mathematics "in statu nascendi," in the process of being invented, has never before been presented in quite this manner to the student, or to the teacher himself, or to the general public.

The subject of heuristic has manifold connections; mathematicians, logicians, psychologists, educationalists, even philosophers may claim various parts of it as belonging to their special domains. The author, well aware of the possibility of criticism from opposite quarters and keenly conscious of his limitations, has one claim to make: he has some experience in solving problems and in teaching mathematics on various levels.

The subject is more fully dealt with in a more extensive book by the author which is on the way to completion.

Stanford University, August 1, 1944

HOW TO SOLVE IT

UNDERSTANDING THE PROBLEM

First.

You have to *understand* the problem.

What is the unknown? What are the data? What is the condition?

Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?

Draw a figure. Introduce suitable notation.

Separate the various parts of the condition. Can you write them down?

DEVISING A PLAN

Second.

Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution.

Have you seen it before? Or have you seen the same problem in a slightly different form?

Do you know a related problem? Do you know a theorem that could be useful?

Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.

Here is a problem related to yours and solved before. Could you use it? Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?

Could you restate the problem? Could you restate it still differently? Go back to definitions.

If you cannot solve the proposed problem try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or the data, or both if necessary, so that the new unknown and the new data are nearer to each other?

Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

CARRYING OUT THE PLAN

Third.

Carry out your plan.

Carrying out your plan of the solution, *check each step*. Can you see clearly that the step is correct? Can you prove that it is correct?

LOOKING BACK

Fourth.

Examine the solution obtained.

Can you *check the result?* Can you check the argument?

Can you derive the result differently? Can you see it at a glance?

Can you use the result, or the method, for some other problem?

PART I. IN THE CLASSROOM

PURPOSE

1. Helping the student. One of the most important tasks of the teacher is to help his students. This task is not quite easy; it demands time, practice, devotion, and sound principles.

The student should acquire as much experience of independent work as possible. But if he is left alone with his problem without any help or with insufficient help, he may make no progress at all. If the teacher helps too much, nothing is left to the student. The teacher should help, but not too much and not too little, so that the student shall have a *reasonable share of the work*.

If the student is not able to do much, the teacher should leave him at least some illusion of independent work. In order to do so, the teacher should help the student discreetly, *unobtrusively*.

The best is, however, to help the student naturally. The teacher should put himself in the student's place, he should see the student's case, he should try to understand what is going on in the student's mind, and ask a question or indicate a step that *could have occurred to the student himself*.

2. Questions, recommendations, mental operations. Trying to help the student effectively but unobtrusively and naturally, the teacher is led to ask the same questions and to indicate the same steps again and again. Thus, in countless problems, we have to ask the question: *What* *is the unknown?* We may vary the words, and ask the same thing in many different ways: What is required? What do you want to find? What are you supposed to seek? The aim of these questions is to focus the student's attention upon the unknown. Sometimes, we obtain the same effect more naturally with a suggestion: *Look at the unknown!* Question and suggestion aim at the same effect; they tend to provoke the same mental operation.

It seemed to the author that it might be worth while to collect and to group questions and suggestions which are typically helpful in discussing problems with students. The list we study contains questions and suggestions of this sort, carefully chosen and arranged; they are equally useful to the problem-solver who works by himself. If the reader is sufficiently acquainted with the list and can see, behind the suggestion, the action suggested, he may realize that the list enumerates, indirectly, *mental operations typically useful for the solution of problems.* These operations are listed in the order in which they are most likely to occur.

3. **Generality** is an important characteristic of the questions and suggestions contained in our list. Take the questions: *What is the unknown? What are the data? What is the condition?* These questions are generally applicable, we can ask them with good effect dealing with all sorts of problems. Their use is not restricted to any subject-matter. Our problem may be algebraic or geometric, mathematical or nonmathematical, theoretical or practical, a serious problem or a mere puzzle; it makes no difference, the questions make sense and might help us to solve the problem.

There is a **restriction,** in fact, but it has nothing to do with the subject-matter. Certain questions and suggestions of the list are applicable to "problems to find" only, not to "problems to prove." If we have a problem of the latter kind we must use different questions; see PROBLEMS TO FIND, PROBLEMS TO PROVE.

4. **Common sense.** The questions and suggestions of our list are general, but, except for their generality, they are natural, simple, obvious, and proceed from plain common sense. Take the suggestion: *Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.* This suggestion advises you to do what you would do anyhow, without any advice, if you were seriously concerned with your problem. Are you hungry? You wish to obtain food and you think of familiar ways of obtaining food. Have you a problem of geometric construction? You wish to construct a triangle and you think of familiar ways of constructing a triangle. Have you a problem of any kind? You wish to find a certain unknown, and you think of familiar ways of finding such an unknown, or some similar unknown. If you do so you follow exactly the suggestion we quoted from our list. And you are on the right track, too; the suggestion is a good one, it suggests to you a procedure which is very frequently successful.

All the questions and suggestions of our list are natural, simple, obvious, just plain common sense; but they state plain common sense in general terms. They suggest a certain conduct which comes naturally to any person who is seriously concerned with his problem and has some common sense. But the person who behaves the right way usually does not care to express his behavior in clear words and, possibly, he cannot express it so; our list tries to express it so.

5. **Teacher and student. Imitation and practice.** There are two aims which the teacher may have in view when addressing to his students a question or a suggestion of the list: First, to help the student to solve the problem

at hand. Second, to develop the student's ability so that he may solve future problems by himself.

Experience shows that the questions and suggestions of our list, appropriately used, very frequently help the student. They have two common characteristics, common sense and generality. As they proceed from plain common sense they very often come naturally; they could have occurred to the student himself. As they are general, they help unobtrusively; they just indicate a general direction and leave plenty for the student to do.

But the two aims we mentioned before are closely connected; if the student succeeds in solving the problem at hand, he adds a little to his ability to solve problems. Then, we should not forget that our questions are general, applicable in many cases. If the same question is repeatedly helpful, the student will scarcely fail to notice it and he will be induced to ask the question by himself in a similar situation. Asking the question repeatedly, he may succeed once in eliciting the right idea. By such a success, he discovers the right way of using the question, and then he has really assimilated it.

The student may absorb a few questions of our list so well that he is finally able to put to himself the right question in the right moment and to perform the corresponding mental operation naturally and vigorously. Such a student has certainly derived the greatest possible profit from our list. What can the teacher do in order to obtain this best possible result?

Solving problems is a practical skill like, let us say, swimming. We acquire any practical skill by imitation and practice. Trying to swim, you imitate what other people do with their hands and feet to keep their heads above water, and, finally, you learn to swim by practicing swimming. Trying to solve problems, you have to observe and to imitate what other people do when solv-

ing problems and, finally, you learn to do problems by doing them.

The teacher who wishes to develop his students' ability to do problems must instill some interest for problems into their minds and give them plenty of opportunity for imitation and practice. If the teacher wishes to develop in his students the mental operations which correspond to the questions and suggestions of our list, he puts these questions and suggestions to the students as often as he can do so naturally. Moreover, when the teacher solves a problem before the class, he should dramatize his ideas a little and he should put to himself the same questions which he uses when helping the students. Thanks to such guidance, the student will eventually discover the right use of these questions and suggestions, and doing so he will acquire something that is more important than the knowledge of any particular mathematical fact.

MAIN DIVISIONS, MAIN QUESTIONS

6. Four phases. Trying to find the solution, we may repeatedly change our point of view, our way of looking at the problem. We have to shift our position again and again. Our conception of the problem is likely to be rather incomplete when we start the work; our outlook is different when we have made some progress; it is again different when we have almost obtained the solution.

In order to group conveniently the questions and suggestions of our list, we shall distinguish four phases of the work. First, we have to *understand* the problem; we have to see clearly what is required. Second, we have to see how the various items are connected, how the unknown is linked to the data, in order to obtain the idea of the solution, to make a *plan*. Third, we *carry out* our plan. Fourth, we *look back* at the completed solution, we review and discuss it.

Each of these phases has its importance. It may happen that a student hits upon an exceptionally bright idea and jumping all preparations blurts out with the solution. Such lucky ideas, of course, are most desirable, but something very undesirable and unfortunate may result if the student leaves out any of the four phases without having a good idea. The worst may happen if the student embarks upon computations or constructions without having *understood* the problem. It is generally useless to carry out details without having seen the main connection, or having made a sort of *plan*. Many mistakes can be avoided if, carrying out his plan, the student *checks each step*. Some of the best effects may be lost if the student fails to reexamine and to *reconsider* the completed solution.

7. Understanding the problem. It is foolish to answer a question that you do not understand. It is sad to work for an end that you do not desire. Such foolish and sad things often happen, in and out of school, but the teacher should try to prevent them from happening in his class. The student should understand the problem. But he should not only understand it, he should also desire its solution. If the student is lacking in understanding or in interest, it is not always his fault; the problem should be well chosen, not too difficult and not too easy, natural and interesting, and some time should be allowed for natural and interesting presentation.

First of all, the verbal statement of the problem must be understood. The teacher can check this, up to a certain extent; he asks the student to repeat the statement, and the student should be able to state the problem fluently. The student should also be able to point out the principal parts of the problem, the unknown, the

data, the condition. Hence, the teacher can seldom afford to miss the questions: *What is the unknown? What are the data? What is the condition?*

The student should consider the principal parts of the problem attentively, repeatedly, and from various sides. If there is a figure connected with the problem he should *draw a figure* and point out on it the unknown and the data. If it is necessary to give names to these objects he should *introduce suitable notation;* devoting some attention to the appropriate choice of signs, he is obliged to consider the objects for which the signs have to be chosen. There is another question which may be useful in this preparatory stage provided that we do not expect a definitive answer but just a provisional answer, a guess: *Is it possible to satisfy the condition?*

(In the exposition of Part II [p. 33] "Understanding the problem" is subdivided into two stages: "Getting acquainted" and "Working for better understanding.")

8. Example. Let us illustrate some of the points explained in the foregoing section. We take the following simple problem: *Find the diagonal of a rectangular parallelepiped of which the length, the width, and the height are known.*

In order to discuss this problem profitably, the students must be familiar with the theorem of Pythagoras, and with some of its applications in plane geometry, but they may have very little systematic knowledge in solid geometry. The teacher may rely here upon the student's unsophisticated familiarity with spatial relations.

The teacher can make the problem interesting by making it concrete. The classroom is a rectangular parallelepiped whose dimensions could be measured, and can be estimated; the students have to find, to "measure indirectly," the diagonal of the classroom. The teacher points out the length, the width, and the height of the classroom, indicates the diagonal with a gesture, and enlivens his figure, drawn on the blackboard, by referring repeatedly to the classroom.

The dialogue between the teacher and the students may start as follows:

"*What is the unknown?*"

"The length of the diagonal of a parallelepiped."

"*What are the data?*"

"The length, the width, and the height of the parallelepiped."

"*Introduce suitable notation.* Which letter should denote the unknown?"

"*x*."

"Which letters would you choose for the length, the width, and the height?"

"*a*, *b*, *c*."

"*What is the condition,* linking *a*, *b*, *c*, and *x*?"

"*x* is the diagonal of the parallelepiped of which *a*, *b*, and *c* are the length, the width, and the height."

"Is it a reasonable problem? I mean, *is the condition sufficient to determine the unknown?*"

"Yes, it is. If we know *a*, *b*, *c*, we know the parallelepiped. If the parallelepiped is determined, the diagonal is determined."

9. Devising a plan. We have a plan when we know, or know at least in outline, which calculations, computations, or constructions we have to perform in order to obtain the unknown. The way from understanding the problem to conceiving a plan may be long and tortuous. In fact, the main achievement in the solution of a problem is to conceive the idea of a plan. This idea may emerge gradually. Or, after apparently unsuccessful trials and a period of hesitation, it may occur suddenly, in a flash, as a "bright idea." The best that the teacher can do for the student is to procure for him, by unobtrusive

help, a bright idea. The questions and suggestions we are going to discuss tend to provoke such an idea.

In order to be able to see the student's position, the teacher should think of his own experience, of his difficulties and successes in solving problems.

We know, of course, that it is hard to have a good idea if we have little knowledge of the subject, and impossible to have it if we have no knowledge. Good ideas are based on past experience and formerly acquired knowledge. Mere remembering is not enough for a good idea, but we cannot have any good idea without recollecting some pertinent facts; materials alone are not enough for constructing a house but we cannot construct a house without collecting the necessary materials. The materials necessary for solving a mathematical problem are certain relevant items of our formerly acquired mathematical knowledge, as formerly solved problems, or formerly proved theorems. Thus, it is often appropriate to start the work with the question: *Do you know a related problem?*

The difficulty is that there are usually too many problems which are somewhat related to our present problem, that is, have some point in common with it. How can we choose the one, or the few, which are really useful? There is a suggestion that puts our finger on an essential common point: *Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.*

If we succeed in recalling a formerly solved problem which is closely related to our present problem, we are lucky. We should try to deserve such luck; we may deserve it by exploiting it. *Here is a problem related to yours and solved before. Could you use it?*

The foregoing questions, well understood and seriously considered, very often help to start the right train of ideas; but they cannot help always, they cannot work magic. If they do not work, we must look around for some other appropriate point of contact, and explore the various aspects of our problem; we have to vary, to transform, to modify the problem. *Could you restate the problem?* Some of the questions of our list hint specific means to vary the problem, as generalization, specialization, use of analogy, dropping a part of the condition, and so on; the details are important but we cannot go into them now. Variation of the problem may lead to some appropriate auxiliary problem: *If you cannot solve the proposed problem try to solve first some related problem.*

Trying to apply various known problems or theorems, considering various modifications, experimenting with various auxiliary problems, we may stray so far from our original problem that we are in danger of losing it altogether. Yet there is a good question that may bring us back to it: *Did you use all the data? Did you use the whole condition?*

10. Example. We return to the example considered in section 8. As we left it, the students just succeeded in understanding the problem and showed some mild interest in it. They could now have some ideas of their own, some initiative. If the teacher, having watched sharply, cannot detect any sign of such initiative he has to resume carefully his dialogue with the students. He must be prepared to repeat with some modification the questions which the students do not answer. He must be prepared to meet often with the disconcerting silence of the students (which will be indicated by dots . . . . .).

"Do you know a related problem?"

. . . . .

"Look at the unknown! Do you know a problem having the same unknown?"

. . . . .

"Well, *what is the unknown?"*

"The diagonal of a parallelepiped."

"Do you know any *problem with the same unknown?"*

"No. We have not had any problem yet about the diagonal of a parallelepiped."

"Do you know any *problem with a similar unknown?"*

. . . . .

"You see, the diagonal is a segment, the segment of a straight line. Did you never solve a problem whose unknown was the length of a line?"

"Of course, we have solved such problems. For instance, to find a side of a right triangle."

"Good! *Here is a problem related to yours and solved before. Could you use it?"*

. . . . .

"You were lucky enough to remember a problem which is related to your present one and which you solved

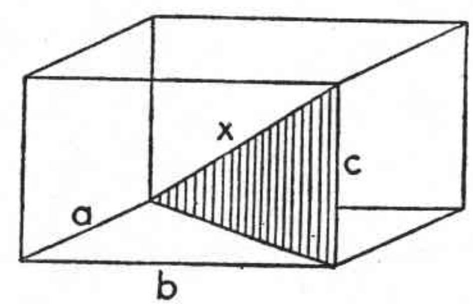

FIG. 1

before. Would you like to use it? *Could you introduce some auxiliary element in order to make its use possible?"*

. . . . .

"Look here, the problem you remembered is about a triangle. Have you any triangle in your figure?"

Let us hope that the last hint was explicit enough to provoke the idea of the solution which is to introduce a right triangle, (emphasized in Fig. 1) of which the required diagonal is the hypotenuse. Yet the teacher should be prepared for the case that even this fairly explicit hint is insufficient to shake the torpor of the students; and so he should be prepared to use a whole gamut of more and more explicit hints.

"Would you like to have a triangle in the figure?"

"What sort of triangle would you like to have in the figure?"

"You cannot find yet the diagonal; but you said that you could find the side of a triangle. Now, what will you do?"

"Could you find the diagonal, if it were a side of a triangle?"

When, eventually, with more or less help, the students succeed in introducing the decisive auxiliary element, the right triangle emphasized in Fig. 1, the teacher should convince himself that the students see sufficiently far ahead before encouraging them to go into actual calculations.

"I think that it was a good idea to draw that triangle. You have now a triangle; but have you the unknown?"

"The unknown is the hypotenuse of the triangle; we can calculate it by the theorem of Pythagoras."

"You can, if both legs are known; but are they?"

"One leg is given, it is c. And the other, I think, is not difficult to find. Yes, the other leg is the hypotenuse of another right triangle."

"Very good! Now I see that you have a plan."

11. Carrying out the plan. To devise a plan, to conceive the idea of the solution is not easy. It takes so much to succeed; formerly acquired knowledge, good mental habits, concentration upon the purpose, and one more thing: good luck. To carry out the plan is much easier; what we need is mainly patience.

The plan gives a general outline; we have to convince

ourselves that the details fit into the outline, and so we have to examine the details one after the other, patiently, till everything is perfectly clear, and no obscure corner remains in which an error could be hidden.

If the student has really conceived a plan, the teacher has now a relatively peaceful time. The main danger is that the student forgets his plan. This may easily happen if the student received his plan from outside, and accepted it on the authority of the teacher; but if he worked for it himself, even with some help, and conceived the final idea with satisfaction, he will not lose this idea easily. Yet the teacher must insist that the student should *check each step*.

We may convince ourselves of the correctness of a step in our reasoning either "intuitively" or "formally." We may concentrate upon the point in question till we see it so clearly and distinctly that we have no doubt that the step is correct; or we may derive the point in question according to formal rules. (The difference between "insight" and "formal proof" is clear enough in many important cases; we may leave further discussion to philosophers.)

The main point is that the student should be honestly convinced of the correctness of each step. In certain cases, the teacher may emphasize the difference between "seeing" and "proving": *Can you see clearly that the step is correct?* But can you also *prove that the step is correct?*

12. Example. Let us resume our work at the point where we left it at the end of section **10**. The student, at last, has got the idea of the solution. He sees the right triangle of which the unknown x is the hypotenuse and the given height c is one of the legs; the other leg is the diagonal of a face. The student must, possibly, be urged to introduce suitable notation. He should choose y to denote that other leg, the diagonal of the face whose sides are a and b. Thus, he may see more clearly the idea of the solution which is to introduce an auxiliary problem whose unknown is y. Finally, working at one right triangle after the other, he may obtain (see Fig. 1)

$$x^2 = y^2 + c^2$$
$$y^2 = a^2 + b^2$$

and hence, eliminating the auxiliary unknown y,

$$x^2 = a^2 + b^2 + c^2$$
$$x = \sqrt{a^2 + b^2 + c^2}\,.$$

The teacher has no reason to interrupt the student if he carries out these details correctly except, possibly, to warn him that he should *check each step*. Thus, the teacher may ask:

"Can you *see clearly* that the triangle with sides x, y, c is a right triangle?"

To this question the student may answer honestly "Yes" but he could be much embarrassed if the teacher, not satisfied with the intuitive conviction of the student, should go on asking:

"But can you *prove* that this triangle is a right triangle?"

Thus, the teacher should rather suppress this question unless the class has had a good initiation in solid geometry. Even in the latter case, there is some danger that the answer to an incidental question may become the main difficulty for the majority of the students.

13. Looking back. Even fairly good students, when they have obtained the solution of the problem and written down neatly the argument, shut their books and look for something else. Doing so, they miss an important and instructive phase of the work. By looking back at the completed solution, by reconsidering and reexamining the result and the path that led to it, they could consolidate their knowledge and develop their ability to solve problems. A good teacher should understand and impress on his students the view that no problem whatever is completely exhausted. There remains always something to do; with sufficient study and penetration, we could improve any solution, and, in any case, we can always improve our understanding of the solution.

The student has now carried through his plan. He has written down the solution, checking each step. Thus, he should have good reasons to believe that his solution is correct. Nevertheless, errors are always possible, especially if the argument is long and involved. Hence, verifications are desirable. Especially, if there is some rapid and intuitive procedure to test either the result or the argument, it should not be overlooked. *Can you check the result? Can you check the argument?*

In order to convince ourselves of the presence or of the quality of an object, we like to see and to touch it. And as we prefer perception through two different senses, so we prefer conviction by two different proofs: *Can you derive the result differently?* We prefer, of course, a short and intuitive argument to a long and heavy one: *Can you see it at a glance?*

One of the first and foremost duties of the teacher is not to give his students the impression that mathematical problems have little connection with each other, and no connection at all with anything else. We have a natural opportunity to investigate the connections of a problem when looking back at its solution. The students will find looking back at the solution really interesting if they have made an honest effort, and have the consciousness of having done well. Then they are eager to see what else they could accomplish with that effort, and how they could do equally well another time. The teacher should encourage the students to imagine cases in which they could utilize again the procedure used, or apply the result obtained. *Can you use the result, or the method, for some other problem?*

14. Example. In section 12, the students finally obtained the solution: If the three edges of a rectangular parallelogram, issued from the same corner, are a, b, c, the diagonal is

$$\sqrt{a^2 + b^2 + c^2}.$$

Can you check the result? The teacher cannot expect a good answer to this question from inexperienced students. The students, however, should acquire fairly early the experience that problems "in letters" have a great advantage over purely numerical problems; if the problem is given "in letters" its result is accessible to several tests to which a problem "in numbers" is not susceptible at all. Our example, although fairly simple, is sufficient to show this. The teacher can ask several questions about the result which the students may readily answer with "Yes"; but an answer "No" would show a serious flaw in the result.

"*Did you use all the data?* Do all the data a, b, c appear in your formula for the diagonal?"

"Length, width, and height play the same role in our question; our problem is symmetric with respect to a, b, c. Is the expression you obtained for the diagonal symmetric in a, b, c? Does it remain unchanged when a, b, c are interchanged?"

"Our problem is a problem of solid geometry: to find the diagonal of a parallelepiped with given dimensions a, b, c. Our problem is analogous to a problem of plane geometry: to find the diagonal of a rectangle with given dimensions a, b. Is the result of our 'solid' problem analogous to the result of the 'plane' problem?"

"If the height c decreases, and finally vanishes, the

parallelepiped becomes a parallelogram. If you put $c = 0$ in your formula, do you obtain the correct formula for the diagonal of the rectangular parallelogram?"

"If the height c increases, the diagonal increases. Does your formula show this?"

"If all three measures a, b, c of the parallelepiped increase in the same proportion, the diagonal also increases in the same proportion. If, in your formula, you substitute $12a$, $12b$, $12c$ for a, b, c respectively, the expression of the diagonal, owing to this substitution, should also be multiplied by 12. Is that so?"

"If a, b, c are measured in feet, your formula gives the diagonal measured in feet too; but if you change all measures into inches, the formula should remain correct. Is that so?"

(The two last questions are essentially equivalent; see TEST BY DIMENSION.)

These questions have several good effects. First, an intelligent student cannot help being impressed by the fact that the formula passes so many tests. He was convinced before that the formula is correct because he derived it carefully. But now he is more convinced, and his gain in confidence comes from a different source; it is due to a sort of "experimental evidence." Then, thanks to the foregoing questions, the details of the formula acquire new significance, and are linked up with various facts. The formula has therefore a better chance of being remembered, the knowledge of the student is consolidated. Finally, these questions can be easily transferred to similar problems. After some experience with similar problems, an intelligent student may perceive the underlying general ideas: use of all relevant data, variation of the data, symmetry, analogy. If he gets into the habit of directing his attention to such points, his ability to solve problems may definitely profit.

Can you check the argument? To recheck the argument step by step may be necessary in difficult and important cases. Usually, it is enough to pick out "touchy" points for rechecking. In our case, it may be advisable to discuss retrospectively the question which was less advisable to discuss as the solution was not yet attained: Can you *prove* that the triangle with sides x, y, c is a right triangle? (See the end of section **12.**)

Can you use the result or the method for some other problem? With a little encouragement, and after one or two examples, the students easily find applications which consist essentially in giving some *concrete interpretation* to the abstract mathematical elements of the problem. The teacher himself used such a concrete interpretation as he took the room in which the discussion takes place for the parallelepiped of the problem. A dull student may propose, as application, to calculate the diagonal of the cafeteria instead of the diagonal of the classroom. If the students do not volunteer more imaginative remarks, the teacher himself may put a slightly different problem, for instance: "Being given the length, the width, and the height of a rectangular parallelepiped, find the distance of the center from one of the corners."

The students may use the *result* of the problem they just solved, observing that the distance required is one half of the diagonal they just calculated. Or they may use the *method,* introducing suitable right triangles (the latter alternative is less obvious and somewhat more clumsy in the present case).

After this application, the teacher may discuss the configuration of the four diagonals of the parallelepiped, and the six pyramids of which the six faces are the bases, the center the common vertex, and the semidiagonals the lateral edges. When the geometric imagination of the students is sufficiently enlivened, the teacher should come

back to his question: *Can you use the result, or the method, for some other problem?* Now there is a better chance that the students may find some more interesting concrete interpretation, for instance, the following:

"In the center of the flat rectangular top of a building which is 21 yards long and 16 yards wide, a flagpole is to be erected, 8 yards high. To support the pole, we need four equal cables. The cables should start from the same point, 2 yards under the top of the pole, and end at the four corners of the top of the building. How long is each cable?"

The students may use the *method* of the problem they solved in detail introducing a right triangle in a vertical plane, and another one in a horizontal plane. Or they may use the *result,* imagining a rectangular parallelepiped of which the diagonal, x, is one of the four cables and the edges are

$$a = 10.5 \qquad b = 8 \qquad c = 6.$$

By straightforward application of the formula, $x = 14.5$.

For more examples, see CAN YOU USE THE RESULT?

15. Various approaches. Let us still retain, for a while, the problem we considered in the foregoing sections **8, 10, 12, 14.** The main work, the discovery of the plan, was described in section **10.** Let us observe that the teacher could have proceeded differently. Starting from the same point as in section **10,** he could have followed a somewhat different line, asking the following questions:

"Do you know any related problem?"

"Do you know an *analogous* problem?"

"You see, the proposed problem is a problem of solid geometry. Could you think of a simpler analogous problem of plane geometry?"

"You see, the proposed problem is about a figure in space, it is concerned with the diagonal of a rectangular parallelepiped. What might be an analogous problem about a figure in the plane? It should be concerned with —the diagonal—of—a rectangular—"

"Parallelogram."

The students, even if they are very slow and indifferent, and were not able to guess anything before, are obliged finally to contribute at least a minute part of the idea. Besides, if the students are so slow, the teacher should not take up the present problem about the parallelepiped without having discussed before, in order to prepare the students, the analogous problem about the parallelogram. Then, he can go on now as follows:

"Here is a problem related to yours and solved before. Can you use it?"

"Should you introduce some auxiliary element in order to make its use possible?"

Eventually, the teacher may succeed in suggesting to the students the desirable idea. It consists in conceiving the diagonal of the given parallelepiped as the diagonal of a suitable parallelogram which must be introduced into the figure (as intersection of the parallelepiped with a plane passing through two opposite edges). The idea is essentially the same as before (section **10**) but the approach is different. In section **10**, the contact with the available knowledge of the students was established through the unknown; a formerly solved problem was recollected because its unknown was the same as that of the proposed problem. In the present section analogy provides the contact with the idea of the solution.

16. The teacher's method of questioning shown in the foregoing sections **8, 10, 12, 14, 15** is essentially this: Begin with a general question or suggestion of our list, and, if necessary, come down gradually to more specific and concrete questions or suggestions till you reach one which elicits a response in the student's mind. If you

have to help the student exploit his idea, start again, if possible, from a general question or suggestion contained in the list, and return again to some more special one if necessary; and so on.

Of course, our list is just a first list of this kind; it seems to be sufficient for the majority of simple cases, but there is no doubt that it could be perfected. It is important, however, that the suggestions from which we start should be simple, natural, and general, and that their list should be short.

The suggestions must be simple and natural because otherwise they cannot be *unobtrusive.*

The suggestions must be general, applicable not only to the present problem but to problems of all sorts, if they are to help develop the *ability* of the student and not just a special technique.

The list must be short in order that the questions may be often repeated, unartificially, and under varying circumstances; thus, there is a chance that they will be eventually assimilated by the student and will contribute to the development of a *mental habit.*

It is necessary to come down gradually to specific suggestions, in order that the student may have as great a *share of the work* as possible.

This method of questioning is not a rigid one; fortunately so, because, in these matters, any rigid, mechanical, pedantical procedure is necessarily bad. Our method admits a certain elasticity and variation, it admits various approaches (section **15**), it can be and should be so applied that questions asked by the teacher *could have occurred to the student himself.*

If a reader wishes to try the method here proposed in his class he should, of course, proceed with caution. He should study carefully the example introduced in section **8**, and the following examples in sections **18, 19, 20**. He should prepare carefully the examples which he intends to discuss, considering also various approaches. He should start with a few trials and find out gradually how he can manage the method, how the students take it, and how much time it takes.

17. Good questions and bad questions. If the method of questioning formulated in the foregoing section is well understood it helps to judge, by comparison, the quality of certain suggestions which may be offered with the intention of helping the students.

Let us go back to the situation as it presented itself at the beginning of section **10** when the question was asked: *Do you know a related problem?* Instead of this, with the best intention to help the students, the question may be offered: *Could you apply the theorem of Pythagoras?*

The intention may be the best, but the question is about the worst. We must realize in what situation it was offered; then we shall see that there is a long sequence of objections against that sort of "help."

(1) If the student is near to the solution, he may understand the suggestion implied by the question; but if he is not, he quite possibly will not see at all the point at which the question is driving. Thus the question fails to help where help is most needed.

(2) If the suggestion is understood, it gives the whole secret away, very little remains for the student to do.

(3) The suggestion is of too special a nature. Even if the student can make use of it in solving the present problem, nothing is learned for future problems. The question is not instructive.

(4) Even if he understands the suggestion, the student can scarcely understand how the teacher came to the idea of putting such a question. And how could he, the student, find such a question by himself? It appears as an unnatural surprise, as a rabbit pulled out of a hat; it is really not instructive.

None of these objections can be raised against the procedure described in section **10**, or against that in section **15**.

MORE EXAMPLES

18. A problem of construction. *Inscribe a square in a given triangle. Two vertices of the square should be on the base of the triangle, the two other vertices of the square on the two other sides of the triangle, one on each.*

"What is the unknown?"

"A square."

"What are the data?"

"A triangle is given, nothing else."

"What is the condition?"

"The four corners of the square should be on the perimeter of the triangle, two corners on the base, one corner on each of the other two sides."

"Is it possible to satisfy the condition?"

"I think so. I am not so sure."

"You do not seem to find the problem too easy. *If you cannot solve the proposed problem, try to solve first some related problem.* Could you satisfy a *part of the condition?*"

"What do you mean by a part of the condition?"

"You see, the condition is concerned with all the vertices of the square. How many vertices are there?"

"Four."

"A part of the condition would be concerned with less than four vertices. *Keep only a part of the condition, drop the other part.* What part of the condition is easy to satisfy?"

"It is easy to draw a square with two vertices on the perimeter of the triangle—or even one with three vertices on the perimeter!"

"Draw a figure!"

The student draws Fig. 2.

"You *kept only a part of the condition,* and you *dropped the other part. How far is the unknown now determined?*"

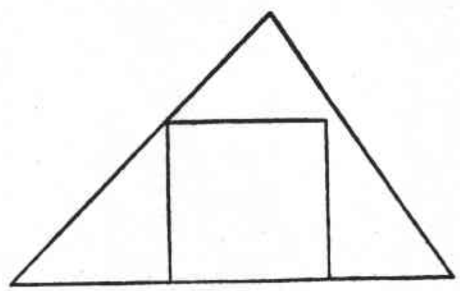

FIG. 2

"The square is not determined if it has only three vertices on the perimeter of the triangle."

"Good! *Draw a figure.*"

The student draws Fig. 3.

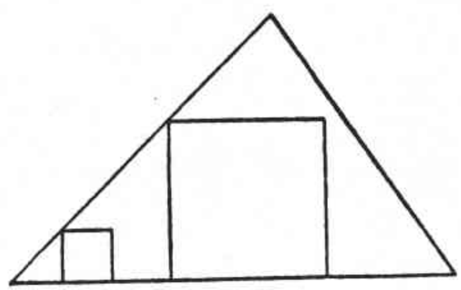

FIG. 3

"The square, as you said, is not determined by the *part of the condition you kept. How can it vary?*"

. . . . .

"Three corners of your square are on the perimeter of the triangle but the fourth corner is not yet there where it should be. Your square, as you said, is undetermined, it can vary; the same is true of its fourth corner. *How can it vary?*"

. . . . .

"Try it experimentally, if you wish. Draw more squares with three corners on the perimeter in the same way as the two squares already in the figure. Draw small squares and large squares. What seems to be the locus of the fourth corner? *How can it vary?*"

The teacher brought the student very near to the idea of the solution. If the student is able to guess that the locus of the fourth corner is a straight line, he has got it.

19. A problem to prove. *Two angles are in different planes but each side of one is parallel to the corresponding side of the other, and has also the same direction. Prove that such angles are equal.*

What we have to prove is a fundamental theorem of solid geometry. The problem may be proposed to students who are familiar with plane geometry and acquainted with those few facts of solid geometry which prepare the present theorem in Euclid's Elements. (The theorem that we have stated and are going to prove is the proposition 10 of Book XI of Euclid.) Not only questions and suggestions quoted from our list are printed in italics but also others which correspond to them as "problems to prove" correspond to "problems to find." (The correspondence is worked out systematically in PROBLEMS TO FIND, PROBLEMS TO PROVE 5, 6.)

"*What is the hypothesis?*"

"Two angles are in different planes. Each side of one is parallel to the corresponding side of the other, and has also the same direction.

"*What is the conclusion?*"

"The angles are equal."

"*Draw a figure. Introduce suitable notation.*"

The student draws the lines of Fig. 4 and chooses, helped more or less by the teacher, the letters as in Fig. 4.

"What is the hypothesis? Say it, please, using your notation."

"A, B, C are not in the same plane as A', B', C'. And $AB \parallel A'B'$, $AC \parallel A'C'$. Also AB has the same direction as $A'B'$, and AC the same as $A'C'$."

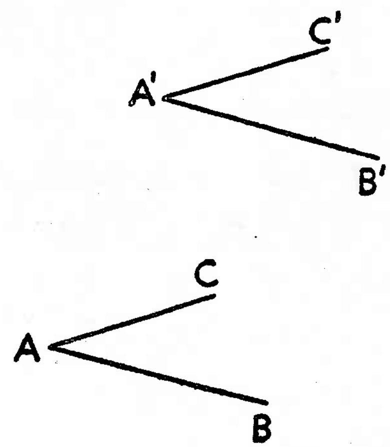

FIG. 4

"What is the conclusion?"

"$\angle BAC = \angle B'A'C'$."

"Look at the conclusion! And try to think of a familiar theorem having the same or a similar conclusion."

"If two triangles are congruent, the corresponding angles are equal."

"Very good! Now *here is a theorem related to yours and proved before. Could you use it?"*

"I think so but I do not see yet quite how."

"Should you introduce some auxiliary element in order to make its use possible?"

. . . . .

"Well, the theorem which you quoted so well is about

triangles, about a pair of congruent triangles. Have you any triangles in your figure?"

"No. But I could introduce some. Let me join B to C, and B' to C'. Then there are two triangles, $\triangle ABC$, $\triangle A'B'C'$."

"Well done. But what are these triangles good for?"

"To prove the conclusion, $\angle BAC = \angle B'A'C'$."

"Good! If you wish to prove this, what kind of triangles do you need?"

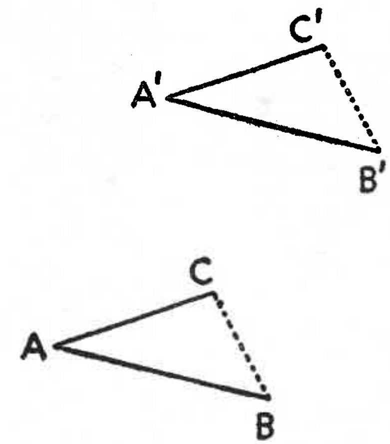

FIG. 5

"Congruent triangles. Yes, of course, I may choose B, C, B', C' so that

$$AB = A'B', AC = A'C'."$$

"Very good! Now, what do you wish to prove?"

"I wish to prove that the triangles are congruent,

$$\triangle ABC = \triangle A'B'C'.$$

If I could prove this, the conclusion $\angle BAC = \angle B'A'C'$ would follow immediately."

"Fine! You have a new aim, you aim at a new conclusion. *Look at the conclusion! And try to think of a* *familiar theorem having the same or a similar conclusion."*

"Two triangles are congruent if—if the three sides of the one are equal respectively to the three sides of the other."

"Well done. You could have chosen a worse one. Now *here is a theorem related to yours and proved before. Could you use it?"*

"I could use it if I knew that $BC = B'C'$."

"That is right! Thus, what is your aim?"

"To prove that $BC = B'C'$."

"Try to think of a familiar theorem having the same or a similar conclusion."

"Yes, I know a theorem finishing: '. . . then the two lines are equal.' But it does not fit in."

"Should you introduce some auxiliary element in order to make its use possible?"

. . . . .

"You see, how could you prove $BC = B'C'$ when there is no connection in the figure between BC and $B'C'$?"

. . . . .

"Did you use the hypothesis? What is the hypothesis?"

"We suppose that $AB \parallel A'B'$, $AC \parallel A'C'$. Yes, of course, I must use that."

"Did you use the whole hypothesis? You say that $AB \parallel A'B'$. Is that all that you know about these lines?"

"No; AB is also equal to $A'B'$, by construction. They are parallel and equal to each other. And so are AC and $A'C'$."

"Two parallel lines of equal length—it is an interesting configuration. *Have you seen it before?"*

"Of course! Yes! Parallelogram! Let me join A to A', B to B', and C to C'."

"The idea is not so bad. How many parallelograms have you now in your figure?"

"Two. No, three. No, two. I mean, there are two of

which you can prove immediately that they are parallelograms. There is a third which seems to be a parallelogram; I hope I can prove that it is one. And then the proof will be finished!"

We could have gathered from his foregoing answers that the student is intelligent. But after this last remark of his, there is no doubt.

This student is able to guess a mathematical result and to distinguish clearly between proof and guess. He knows also that guesses can be more or less plausible. Really, he did profit something from his mathematics classes; he has some real experience in solving problems, he can conceive and exploit a good idea.

20. A rate problem. *Water is flowing into a conical vessel at the rate* r. *The vessel has the shape of a right circular cone, with horizontal base, the vertex pointing downwards; the radius of the base is* a, *the altitude of the*

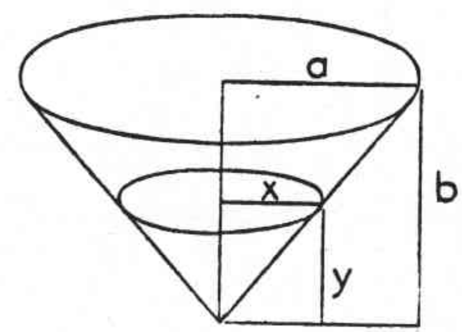

FIG. 6

cone b. *Find the rate at which the surface is rising when the depth of the water is* y. *Finally, obtain the numerical value of the unknown supposing that* a = 4 *ft.*, b = 3 *ft.*, r = 2 *cu. ft. per minute, and* y = 1 *ft.*

The students are supposed to know the simplest rules of differentiation and the notion of "rate of change."

"What are the data?"

"The radius of the base of the cone $a = 4$ ft., the altitude of the cone $b = 3$ ft., the rate at which the water is flowing into the vessel $r = 2$ cu. ft. per minute, and the depth of the water at a certain moment, $y = 1$ ft."

"Correct. The statement of the problem seems to suggest that you should disregard, provisionally, the numerical values, work with the letters, express the unknown in terms of a, b, r, y and only finally, after having obtained the expression of the unknown in letters, substitute the numerical values. I would follow this suggestion. Now, *what is the unknown?*"

"The rate at which the surface is rising when the depth of the water is y."

"What is that? Could you say it in other terms?"

"The rate at which the depth of the water is increasing."

"What is that? *Could you restate it still differently?*"

"The rate of change of the depth of the water."

"That is right, the rate of change of y. But what is the rate of change? *Go back to the definition.*"

"The derivative is the rate of change of a function."

"Correct. Now, is y a function? As we said before, we disregard the numerical value of y. Can you imagine that y changes?"

"Yes, y, the depth of the water, increases as the time goes by."

"Thus, y is a function of what?"

"Of the time t."

"Good. *Introduce suitable notation.* How would you write the 'rate of change of y' in mathematical symbols?"

"$\dfrac{dy}{dt}$"

"Good. Thus, this is your unknown. You have to express it in terms of a, b, r, y. By the way, one of these data is a 'rate.' Which one?"

"r is the rate at which water is flowing into the vessel."

"What is that? Could you say it in other terms?"

"r is the rate of change of the volume of the water in the vessel."

"What is that? *Could you restate it still differently?* How would you write it in *suitable notation?*"

"$r = \dfrac{dV}{dt}$."

"What is V?"

"The volume of the water in the vessel at the time t."

"Good. Thus, you have to express $\dfrac{dy}{dt}$ in terms of $a, b, \dfrac{dV}{dt}, y$. How will you do it?"

. . . . .

"*If you cannot solve the proposed problem try to solve first some related problem.* If you do not see yet the connection between $\dfrac{dy}{dt}$ and the data, try to bring in some simpler connection that could serve as a stepping stone."

. . . . .

"Do you not see that there are other connections? For instance, are y and V independent of each other?"

"No. When y increases, V must increase too."

"Thus, there is a connection. What is the connection?"

"Well, V is the volume of a cone of which the altitude is y. But I do not know yet the radius of the base."

"You may consider it, nevertheless. Call it something, say x."

"$V = \dfrac{\pi x^2 y}{3}$."

"Correct. Now, what about x? Is it independent of y?"

"No. When the depth of the water, y, increases the radius of the free surface, x, increases too."

"Thus, there is a connection. What is the connection?"

"Of course, similar triangles.

$$x : y = a : b."$$

"One more connection, you see. I would not miss profiting from it. Do not forget, you wished to know the connection between V and y."

"I have

$$x = \frac{ay}{b}$$

$$V = \frac{\pi a^2 y^3}{3b^2}."$$

"Very good. This looks like a stepping stone, does it not? But you should not forget your goal. *What is the unknown?*"

"Well, $\frac{dy}{dt}$."

"You have to find a connection between $\frac{dy}{dt}$, $\frac{dV}{dt}$, and other quantities. And here you have one between y, V, and other quantities. What to do?"

"Differentiate! Of course!

$$\frac{dV}{dt} = \frac{\pi a^2 y^2}{b^2}\frac{dy}{dt}.$$

Here it is."

"Fine! And what about the numerical values?"

"If $a = 4$, $b = 3$, $\frac{dV}{dt} = r = 2$, $y = 1$, then

$$2 = \frac{\pi \times 16 \times 1}{9}\frac{dy}{dt}."$$

PART II. HOW TO SOLVE IT A DIALOGUE

Getting Acquainted

Where should I start? Start from the statement of the problem.

What can I do? Visualize the problem as a whole as clearly and as vividly as you can. Do not concern yourself with details for the moment.

What can I gain by doing so? You should understand the problem, familiarize yourself with it, impress its purpose on your mind. The attention bestowed on the problem may also stimulate your memory and prepare for the recollection of relevant points.

Working for Better Understanding

Where should I start? Start again from the statement of the problem. Start when this statement is so clear to you and so well impressed on your mind that you may lose sight of it for a while without fear of losing it altogether.

What can I do? Isolate the principal parts of your problem. The hypothesis and the conclusion are the principal parts of a "problem to prove"; the unknown, the data, and the conditions are the principal parts of a "problem to find." Go through the principal parts of your problem, consider them one by one, consider them in turn, consider them in various combinations, relating each detail to other details and each to the whole of the problem.

What can I gain by doing so? You should prepare and clarify details which are likely to play a role afterwards.

Hunting for the Helpful Idea

Where should I start? Start from the consideration of the principal parts of your problem. Start when these principal parts are distinctly arranged and clearly conceived, thanks to your previous work, and when your memory seems responsive.

What can I do? Consider your problem from various sides and seek contacts with your formerly acquired knowledge.

Consider your problem from various sides. Emphasize different parts, examine different details, examine the same details repeatedly but in different ways, combine the details differently, approach them from different sides. Try to see some new meaning in each detail, some new interpretation of the whole.

Seek contacts with your formerly acquired knowledge. Try to think of what helped you in similar situations in the past. Try to recognize something familiar in what you examine, try to perceive something useful in what you recognize.

What could I perceive? A helpful idea, perhaps a decisive idea that shows you at a glance the way to the very end.

How can an idea be helpful? It shows you the whole of the way or a part of the way; it suggests to you more or less distinctly how you can proceed. Ideas are more or less complete. You are lucky if you have any idea at all.

What can I do with an incomplete idea? You should consider it. If it looks advantageous you should consider it longer. If it looks reliable you should ascertain how far it leads you, and reconsider the situation. The situation has changed, thanks to your helpful idea. Consider the new situation from various sides and seek contacts with your formerly acquired knowledge.

What can I gain by doing so again? You may be lucky and have another idea. Perhaps your next idea will lead you to the solution right away. Perhaps you need a few more helpful ideas after the next. Perhaps you will be led astray by some of your ideas. Nevertheless you should be grateful for all new ideas, also for the lesser ones, also for the hazy ones, also for the supplementary ideas adding some precision to a hazy one, or attempting the correction of a less fortunate one. Even if you do not have any appreciable new ideas for a while you should be grateful if your conception of the problem becomes more complete or more coherent, more homogeneous or better balanced.

Carrying Out the Plan

Where should I start? Start from the lucky idea that led you to the solution. Start when you feel sure of your grasp of the main connection and you feel confident that you can supply the minor details that may be wanting.

What can I do? Make your grasp quite secure. Carry through in detail all the algebraic or geometric operations which you have recognized previously as feasible. Convince yourself of the correctness of each step by formal reasoning, or by intuitive insight, or both ways if you can. If your problem is very complex you may distinguish "great" steps and "small" steps, each great step being composed of several small ones. Check first the great steps, and get down to the smaller ones afterwards.

What can I gain by doing so? A presentation of the solution each step of which is correct beyond doubt.

Looking Back

Where should I start? From the solution, complete and correct in each detail.

What can I do? Consider the solution from various sides and seek contacts with your formerly acquired knowledge.

Consider the details of the solution and try to make them as simple as you can; survey more extensive parts of the solution and try to make them shorter; try to see the whole solution at a glance. Try to modify to their advantage smaller or larger parts of the solution, try to improve the whole solution, to make it intuitive, to fit it into your formerly acquired knowledge as naturally as possible. Scrutinize the method that led you to the solution, try to see its point, and try to make use of it for other problems. Scrutinize the result and try to make use of it for other problems.

What can I gain by doing so? You may find a new and better solution, you may discover new and interesting facts. In any case, if you get into the habit of surveying and scrutinizing your solutions in this way, you will acquire some knowledge well ordered and ready to use, and you will develop your ability of solving problems.

PART III. SHORT DICTIONARY OF HEURISTIC

Analogy is a sort of similarity. Similar objects agree with each other in some respect, analogous objects *agree in certain relations* of their respective parts.

1. A rectangular parallelogram is analogous to a rectangular parallelepiped. In fact, the relations between the sides of the parallelogram are similar to those between the faces of the parallelepiped:

Each side of the parallelogram is parallel to just one other side, and is perpendicular to the remaining sides.

Each face of the parallelepiped is parallel to just one other face, and is perpendicular to the remaining faces.

Let us agree to call a side a "bounding element" of the parallelogram and a face a "bounding element" of the parallelepiped. Then, we may contract the two foregoing statements into one that applies equally to both figures:

Each bounding element is parallel to just one other bounding element and is perpendicular to the remaining bounding elements.

Thus, we have expressed certain relations which are common to the two systems of objects we compared, sides of the rectangle and faces of the rectangular parallelepiped. The analogy of these systems consists in this community of relations.

2. Analogy pervades all our thinking, our everyday speech and our trivial conclusions as well as artistic ways of expression and the highest scientific achievements. Analogy is used on very different levels. People

For another example, a different viewpoint, and further comments see WORKING BACKWARDS.

Compare also REDUCTIO AD ABSURDUM AND INDIRECT PROOF, 2.

Pedantry and mastery are opposite attitudes toward rules.

1. To apply a rule to the letter, rigidly, unquestioningly, in cases where it fits and in cases where it does not fit, is pedantry. Some pedants are poor fools; they never did understand the rule which they apply so conscientiously and so indiscriminately. Some pedants are quite successful; they understood their rule, at least in the beginning (before they became pedants), and chose a good one that fits in many cases and fails only occasionally.

To apply a rule with natural ease, with judgment, noticing the cases where it fits, and without ever letting the words of the rule obscure the purpose of the action or the opportunities of the situation, is mastery.

2. The questions and suggestions of our list may be helpful both to problem-solvers and to teachers. But, first, they must be understood, their proper use must be learned, and learned by trial and error, by failure and success, by experience in applying them. Second, their use should never become pedantic. You should ask no question, make no suggestion, indiscriminately, following some rigid habit. Be prepared for various questions and suggestions and use your judgment. You are doing a hard and exciting problem; the step you are going to try next should be prompted by an attentive and open-minded consideration of the problem before you. You wish to help a student; what you say to your student should proceed from a sympathetic understanding of his difficulties.

And if you are inclined to be a pedant and must rely upon some rule learn this one: Always use your own brains first.

Practical problems are different in various respects from purely mathematical problems, yet the principal motives and procedures of the solution are essentially the same. Practical engineering problems usually involve mathematical problems. We will say a few words about the differences, analogies, and connections between these two sorts of problems.

1. An impressive practical problem is the construction of a dam across a river. We need no special knowledge to understand this problem. In almost prehistoric times, long before our modern age of scientific theories, men built dams of some sort in the valley of the Nile, and in other parts of the world, where the crops depended on irrigation.

Let us visualize the problem of constructing an important modern dam.

What is the unknown? Many unknowns are involved in a problem of this kind: the exact location of the dam, its geometric shape and dimensions, the materials used in its construction, and so on.

What is the condition? We cannot answer this question in one short sentence because there are many conditions. In so large a project it is necessary to satisfy many important economic needs and to hurt other needs as little as possible. The dam should provide electric power, supply water for irrigation or the use of certain communities, and also help to control floods. On the other hand, it should disturb as little as possible navigation, or economically important fish-life, or beautiful scenery; and so forth. And, of course, it should cost as little as possible and be constructed as quickly as possible.

What are the data? The multitude of desirable data is tremendous. We need topographical data concerning the vicinity of the river and its tributaries; geological data important for the solidity of foundations, possible leakage, and available materials of construction; meteorological data about annual precipitation and the height of floods; economic data concerning the value of ground which will be flooded, cost of materials and labor; and so on.

Our example shows that unknowns, data, and conditions are more complex and less sharply defined in a practical problem than in a mathematical problem.

2. In order to solve a problem, we need a certain amount of previously acquired knowledge. The modern engineer has a highly specialized body of knowledge at his disposal, a scientific theory of the strength of materials, his own experience, and the mass of engineering experience stored in special technical literature. We cannot avail ourselves of such special knowledge here but we may try to imagine what was in the mind of an ancient Egyptian dam-builder.

He has seen, of course, various other, perhaps smaller, dams: banks of earth or masonry holding back the water. He has seen the flood, laden with all sorts of debris, pressing against the bank. He might have helped to repair the cracks and the erosion left by the flood. He might have seen a dam break, giving way under the impact of the flood. He has certainly heard stories about dams withstanding the test of centuries or causing catastrophe by an unexpected break. His mind may have pictured the pressure of the river against the surface of the dam and the strain and stress in its interior.

Yet the Egyptian dam-builder had no precise, quantitative, scientific concepts of fluid pressure or of strain and stress in a solid body. Such concepts form an essential

part of the intellectual equipment of a modern engineer. Yet the latter also uses much knowledge which has not yet quite reached a precise, scientific level; what he knows about erosion by flowing water, the transportation of silt, the plasticity and other not quite clearly circumscribed properties of certain materials, is knowledge of a rather empirical character.

Our example shows that the knowledge needed and the concepts used are more complex and less sharply defined in practical problems than in mathematical problems.

3. Unknowns, data, conditions, concepts, necessary preliminary knowledge, everything is more complex and less sharp in practical problems than in purely mathematical problems. This is an important difference, perhaps the main difference, and it certainly implies further differences; yet the fundamental motives and procedures of the solution appear to be the same for both sorts of problems.

There is a widespread opinion that practical problems need more experience than mathematical problems. This may be so. Yet, very likely, the difference lies in the nature of the knowledge needed and not in our attitude toward the problem. In solving a problem of one or the other kind, we have to rely on our experience with similar problems and we often ask the questions: *Have you seen the same problem in a slightly different form? Do you know a related problem?*

In solving a mathematical problem, we start from very clear concepts which are fairly well ordered in our mind. In solving a practical problem, we are often obliged to start from rather hazy ideas; then, the clarification of the concepts may become an important part of the problem. Thus, medical science is in a better position to check infectious diseases today than it was in the times before Pasteur when the notion of infection itself was rather hazy. *Have you taken into account all essential notions involved in the problem?* This is a good question for all sorts of problems but its use varies widely with the nature of the intervening notions.

In a perfectly stated mathematical problem all data and all clauses of the condition are essential and must be taken into account. In practical problems we have a multitude of data and conditions; we take into account as many as we can but we are obliged to neglect some. Take the case of the designer of a large dam. He considers the public interest and important economic interests but he is bound to disregard certain petty claims and grievances. The data of his problem are, strictly speaking, inexhaustible. For instance, he would like to know a little more about the geologic nature of the ground on which the foundations must be laid, but eventually he must stop collecting geologic data although a certain margin of uncertainty unavoidably remains.

Did you use all the data? Did you use the whole condition? We cannot miss these questions when we deal with purely mathematical problems. In practical problems, however, we should put these questions in a modified form: Did you use all the data which *could contribute appreciably* to the solution? Did you use all the conditions which *could influence appreciably* the solution? We take stock of the available relevant information, we collect more information if necessary, but eventually we must stop collecting, we must draw the line somewhere, we cannot help neglecting something. "If you will sail without danger, you must never put to sea." Quite often, there is a great surplus of data which have no appreciable influence on the final form of the solution.

4. The designers of the ancient Egyptian dams had to rely on the common-sense interpretation of their experience, they had nothing else to rely on. The modern engineer cannot rely on common sense alone, especially if his project is of a new and daring design; he has to calculate the resistance of the projected dam, foresee quantitatively the strain and stress in its interior. For this purpose, he has to apply the theory of elasticity (which applies fairly well to constructions in concrete). To apply this theory, he needs a good deal of mathematics; the practical engineering problem leads to a mathematical problem.

This mathematical problem is too technical to be discussed here; all we can say about it is a general remark. In setting up and in solving mathematical problems derived from practical problems, we usually content ourselves with an *approximation.* We are bound to neglect some minor data and conditions of the practical problem. Therefore it is reasonable to allow some slight inaccuracy in the computations especially when we can gain in simplicity what we lose in accuracy.

5. Much could be said about approximations that would deserve general interest. We cannot suppose, however, any specialized mathematical knowledge and therefore we restrict ourselves to just one intuitive and instructive example.

The drawing of geographic maps is an important practical problem. Devising a map, we often assume that the earth is a sphere. Now this is only an approximate assumption and not the exact truth. The surface of the earth is not at all a mathematically defined surface and we definitely know that the earth is flattened at the poles. Assuming, however, that the earth is a sphere, we may draw a map of it much more easily. We gain much in simplicity and do not lose a great deal in accuracy. In fact, let us imagine a big ball that has exactly the shape of the earth and that has a diameter of 25 feet at its equator. The distance between the poles of such a ball is less than 25 feet because the earth is flattened, but only about one inch less. Thus the sphere yields a good practical approximation.

Problems to find, problems to prove. We draw a parallel between these two kinds of problems.

1. The aim of a "problem to find" is to find a certain object, the unknown of the problem.

The unknown is also called "quaesitum," or the thing sought, or the thing required. "Problems to find" may be theoretical or practical, abstract or concrete, serious problems or mere puzzles. We may seek all sorts of unknowns; we may try to find, to obtain, to acquire, to produce, or to construct all imaginable kinds of objects. In the problem of the mystery story the unknown is a murderer. In a chess problem the unknown is a move of the chessmen. In certain riddles the unknown is a word. In certain elementary problems of algebra the unknown is a number. In a problem of geometric construction the unknown is a figure.

2. The aim of a "problem to prove" is to show conclusively that a certain clearly stated assertion is true, or else to show that it is false. We have to answer the question: Is this assertion true or false? And we have to answer conclusively, either by proving the assertion true, or by proving it false.

A witness affirms that the defendant stayed at home a certain night. The judge has to find out whether this assertion is true or not and, moreover, he has to give as good grounds as possible for his finding. Thus, the judge has a "problem to prove." Another "problem to prove" is to "prove the theorem of Pythagoras." We do not say: "Prove or disprove the theorem of Pythagoras." It would be better in some respects to include in the statement of

the problem the possibility of disproving, but we may neglect it, because we know that the chances for disproving the theorem of Pythagoras are rather slight.

3. The principal parts of a "problem to find" are the *unknown*, the *data*, and the *condition*.

If we have to construct a triangle with sides a, b, c, the unknown is a triangle, the data are the three lengths a, b, c, and the triangle is required to satisfy the condition that its sides have the given lengths a, b, c. If we have to construct a triangle whose altitudes are a, b, c, the unknown is an object of the same category as before, the data are the same, but the condition linking the unknown to the data is different.

4. If a "problem to prove" is a mathematical problem of the usual kind, its principal parts are the *hypothesis* and the *conclusion* of the theorem which has to be proved or disproved.

"If the four sides of a quadrilateral are equal, then the two diagonals are perpendicular to each other." The second part starting with "then" is the conclusion, the first part starting with "if" is the hypothesis.

[Not all mathematical theorems can be split naturally into hypothesis and conclusion. Thus, it is scarcely possible to split so the theorem: "There are an infinity of prime numbers."]

5. If you wish to solve a "problem to find" you must know, and know very exactly, its principal parts, the unknown, the data, and the condition. Our list contains many questions and suggestions concerned with these parts.

What is the unknown? What are the data? What is the condition?

Separate the various parts of the condition.

Find the connection between the data and the unknown.

Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.

Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown, or the data, or both if necessary, so that the new unknown and the new data are nearer to each other?

Did you use all the data? Did you use the whole condition?

6. If you wish to solve a "problem to prove" you must know, and know very exactly, its principal parts, the hypothesis, and the conclusion. There are useful questions and suggestions concerning these parts which correspond to those questions and suggestions of our list which are specially adapted to "problems to find."

What is the hypothesis? What is the conclusion?

Separate the various parts of the hypothesis.

Find the connection between the hypothesis and the conclusion.

Look at the conclusion! And try to think of a familiar theorem having the same or a similar conclusion.

Keep only a part of the hypothesis, drop the other part; is the conclusion still valid? Could you derive something useful from the hypothesis? Could you think of another hypothesis from which you could easily derive the conclusion? Could you change the hypothesis, or the conclusion, or both if necessary, so that the new hypothesis and the new conclusion are nearer to each other?

Did you use the whole hypothesis?

7. "Problems to find" are more important in elementary mathematics, "problems to prove" more important in advanced mathematics. In the present book, "problems to find" are more emphasized than the other kind, but the author hopes to reestablish the balance in a fuller treatment of the subject.

Progress and achievement. Have you made any progress? What was the essential achievement? We may address questions of this kind to ourselves when we are solving a problem or to a student whose work we supervise. Thus, we are used to judge, more or less confidently, progress and achievement in concrete cases. The step from such concrete cases to a general description is not easy at all. Yet we have to undertake this step if we wish to make our study of heuristic somewhat complete and we must try to clarify what constitutes, in general, progress and achievement in solving problems.

1. In order to solve a problem, we must have some knowledge of the subject-matter and we must select and collect the relevant items of our existing but initially dormant knowledge. There is much more in our conception of the problem at the end than was in it at the outset; what has been added? What we have succeeded in extracting from our memory. In order to obtain the solution we have to recall various essential facts. We have to recollect formerly solved problems, known theorems, definitions, if our problem is mathematical. Extracting such relevant elements from our memory may be termed *mobilization*.

2. In order to solve a problem, however, it is not enough to recollect isolated facts, we must combine these facts, and their combination must be well adapted to the problem at hand. Thus, in solving a mathematical problem, we have to construct an argument connecting the materials recollected to a well adapted whole. This adapting and combining activity may be termed *organization*.

3. In fact, mobilization and organization can never be

really separated. Working at the problem with concentration, we recall only facts which are more or less connected with our purpose, and we have nothing to connect and organize but materials we have recollected and mobilized.

Mobilization and organization are but two *aspects* of the same complex process which has still many other aspects.

4. Another aspect of the progress of our work is that our *mode of conception changes.* Enriched with all the materials which we have recalled, adapted to it, and worked into it, our conception of the problem is much fuller at the end than it was at the outset. Desiring to proceed from our initial conception of the problem to a more adequate, better adapted conception, we try various standpoints and view the problem from different sides. We could make hardly any progress without VARIATION OF THE PROBLEM.

5. As we progress toward our final goal we see more and more of it, and when we see it better we judge that we are nearer to it. As our examination of the problem advances, we *foresee* more and more clearly what should be done for the solution and how it should be done. Solving a mathematical problem we may foresee, if we are lucky, that a certain known theorem might be used, that the consideration of a certain formerly solved problem might be helpful, that going back to the meaning of a certain technical term might be necessary. We do not foresee such things with certainty, only with a certain degree of plausibility. We shall attain complete certainty when we have obtained the complete solution, but before obtaining certainty we must often be satisfied with a more or less plausible guess. Without considerations which are only plausible and provisional, we could never find the solution which is certain and final. We need HEURISTIC REASONING.

6. What is progress toward the solution? Advancing mobilization and organization of our knowledge, evolution of our conception of the problem, increasing prevision of the steps which will constitute the final argument. We may advance steadily, by small imperceptible steps, but now and then we advance abruptly, by leaps and bounds. A sudden advance toward the solution is called a BRIGHT IDEA, a good idea, a happy thought, a brain-wave (in German there is a more technical term, *Einfall*). What is a bright idea? An abrupt and momentous change of our outlook, a sudden reorganization of our mode of conceiving the problem, a just emerging confident prevision of the steps we have to take in order to attain the solution.

7. The foregoing considerations provide the questions and suggestions of our list with the right sort of background.

Many of these questions and suggestions aim directly at the *mobilization* of our formerly acquired knowledge: *Have you seen it before? Or have you seen the same problem in a slightly different form? Do you know a related problem? Do you know a theorem that could be useful? Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.*

There are typical situations in which we think that we have collected the right sort of material and we work for a better *organization* of what we have mobilized: *Here is a problem related to yours and solved before. Could you use it? Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?*

There are other typical situations in which we think that we have not yet collected enough material. We wonder what is missing: *Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?*

Some questions aim directly at the *variation* of the problem: *Could you restate the problem? Could you restate it still differently?* Many questions aim at the variation of the problem by specified means, as going back to the DEFINITION, using ANALOGY, GENERALIZATION, SPECIALIZATION, DECOMPOSING AND RECOMBINING.

Still other questions suggest a trial to *foresee* the nature of the solution we are striving to obtain: *Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?*

The questions and suggestions of our list do not mention directly the *bright idea*; but, in fact, all are concerned with it. Understanding the problem we prepare for it, devising a plan we try to provoke it, having provoked it we carry it through, looking back at the course and the result of the solution we try to exploit it better.[8]

Puzzles. According to section 3, the questions and suggestions of our list are independent of the subject-matter and applicable to all kinds of problems. It is quite interesting to test this assertion on various puzzles.

Take, for instance, the words

DRY OXTAIL IN REAR.

The problem is to find an "anagram," that is, a rearrangement of the letters contained in the given words into one word. It is interesting to observe that, when we are solving this puzzle, several questions of our list are pertinent and even stimulating.

What is the unknown? A word.

What are the data? The four words DRY OXTAIL IN REAR.

[8] Several points discussed in this article are more fully considered in the author's paper, *Acta Psychologica,* vol. 4 (1938), pp. 113-170.

What is the condition? The desired word has fifteen letters, the letters contained in the four given words. It is probably a not too unusual English word.

Draw a figure. It is quite useful to mark out fifteen blank spaces:

- - - - - - - - - - - - - - -

Could you restate the problem? We have to find a word containing, in some arrangement, the letters

A A E I I O Y   D L N R R R T X.

This is certainly an equivalent restatement of the problem (see AUXILIARY PROBLEM, 6). It may be an advantageous restatement. Separating the vowels from the consonants (this is important, the alphabetical order is not) we see another aspect of the problem. Thus, we see now that the desired word has seven syllables unless it has some diphthongs.

If you cannot solve the proposed problem, try to solve first some related problem. A related problem is to form words with some of the given letters. We can certainly form short words of this kind. Then we try to find longer and longer words. The more letters we use the nearer we may come to the desired word.

Could you solve a part of the problem? The desired word is so long that it must have distinct parts. It is, probably, a compound word, or it is derived from some other word by adding some usual ending. Which usual ending could it be?

- - - - - - - - - - A T I O N
- - - - - - - - - - - - E L Y

Keep only a part of the condition and drop the other part. We may try to think of a long word with, possibly, as many as seven syllables and relatively few consonants, containing an X and a Y.

Signs of progress. As Columbus and his companions sailed westward across an unknown ocean they were cheered whenever they saw birds. They regarded a bird as a favorable sign, indicating the nearness of land. But in this they were repeatedly disappointed. They watched for other signs too. They thought that floating seaweed or low banks of cloud might indicate land, but they were again disappointed. One day, however, the signs multiplied. On Thursday, the 11th of October, 1492, "they saw sandpipers, and a green reed near the ship. Those of the caravel *Pinta* saw a cane and a pole, and they took up another small pole which appeared to have been worked by iron; also another bit of cane, a land-plant, and a small board. The crew of the caravel *Niña* also saw signs of land, and a small branch covered with berries. Everyone breathed afresh and rejoiced at these signs." And in fact the next day they sighted land, the first island of a New World.

Our undertaking may be important or unimportant, our problem of any kind—when we are working intensely, we watch eagerly for signs of progress as Columbus and his companions watched for signs of approaching land. We shall discuss a few examples in order to understand what can be reasonably regarded as a sign of approaching the solution.

1. *Examples.* I have a chess problem. I have to mate the black king in, say, two moves. On the chessboard there is a white knight, quite a distance from the black king, that is apparently superfluous. What is it good for? I am obliged to leave this question unanswered at first. Yet after various trials, I hit upon a new move and observe that it would bring that apparently superfluous white knight into play. This observation gives me a new hope. I regard it as a favorable sign: that new move has some chance to be the right one. Why?

In a well-constructed chess problem there is no superfluous piece. Therefore, we have to take into account all chessmen on the board; we have to *use all the data.* The correct solution does certainly use all the pieces, even that apparently superfluous white knight. In this last respect, the new move that I contemplate agrees with the correct move that I am supposed to find. The new move looks like the correct move; it might be the correct move.

It is interesting to consider a similar situation in a mathematical problem. My task is to express the area of a triangle in terms of its three sides, a, b, and c. I have already made some sort of plan. I know, more or less clearly, which geometrical connections I have to take into account and what sort of calculations I have to perform. Yet I am not quite sure whether my plan will work. If now, proceeding along the line prescribed by my plan, I observe that the quantity

$$\sqrt{b+c-a}$$

enters into the expression of the area I am about to construct, I have good reason to be cheered. Why?

In fact, it must be taken into account that the sum of any two sides of a triangle is greater than the third side. This involves a certain restriction. The given lengths, a, b, and c cannot be quite arbitrary; for instance, $b+c$ must be greater than a. This is an essential part of the condition, and we should *use the whole condition.* If $b+c$ is not greater than a the formula I seek is bound to become illusory. Now, the square root displayed above becomes imaginary if $b+c-a$ is negative—that is, if $b+c$ is less than a—and so the square root becomes unfit to represent a real quantity under just those circumstances under which the desired expression is bound to become illusory. Thus my formula, into which that square root enters, has an important property in common with the true formula for the area. My formula looks like the true formula; it might be the true formula.

Here is one more example. Some time ago, I wished to prove a theorem in solid geometry. Without much trouble I found a first remark that appeared to be pertinent; but then I got stuck. Something was missing to finish the proof. When I gave up that day I had a much clearer notion than at the outset how the proof should look, how the gap should be filled; but I was not able to fill it. The next day, after a good night's rest, I looked again into the question and soon hit upon an analogous theorem in plane geometry. In a flash I was convinced that now I had got hold of the solution and I had, I think, good reason too to be convinced. Why?

In fact, *analogy* is a great guide. The solution of a problem in solid geometry often depends on an analogous problem in plane geometry (see ANALOGY, 3-7). Thus, in my case, there was a chance from the outset that the desired proof would use as a lemma some theorem of plane geometry of the kind which actually came to my mind. "This theorem looks like the lemma I need; it might be the lemma I need"—such was my reasoning.

If Columbus and his men had taken the trouble to reason explicitly, they would have reasoned in some similar way. They knew how the sea looks near the shore. They knew that, more often than on the open sea, there are birds in the air, coming from the land, and objects floating in the water, detached from the seashore. Many of the men must have observed such things when from former voyages they had returned to their home port. The day before that memorable date on which they sighted the island of San Salvador, as the floating objects in the water became so frequent, they thought: "It looks as if we were approaching some land; we may be approaching some land" and "everyone breathed afresh and rejoiced at these signs."

2. *Heuristic character of signs of progress.* Let us insist upon a point which is perhaps already clear to everyone; but it is very important and, therefore, it should be completely clear.

The type of reasoning illustrated by the foregoing examples deserves to be noticed and taken into account seriously, although it yields only a plausible indication and not an unfailing certainty. Let us restate pedantically, at full length, in rather unnatural detail, one of these reasonings:

If we are approaching land, we often see birds.
Now we see birds.
Therefore, probably, we are approaching land.

Without the word "probably" the conclusion would be an outright fallacy. In fact, Columbus and his companions saw birds many times but were disappointed later. Just once came the day on which they saw sandpipers followed by the day of discovery.

With the word "probably" the conclusion is reasonable and natural but by no means a proof, a demonstrative conclusion; it is only an indication, a heuristic suggestion. It would be a great mistake to forget that such a conclusion is only probable, and to regard it as certain. But to disregard such conclusions entirely would be a still greater mistake. If you take a heuristic conclusion as certain, you may be fooled and disappointed; but if you neglect heuristic conclusions altogether you will make no progress at all. The most important signs of progress are heuristic. Should we trust them? Should we follow them? Follow, but keep your eyes open. Trust but look. And never renounce your judgment.

3. *Clearly expressible signs.* We can look at the foregoing examples from another point of view.

In one of these examples, we regarded as a favorable sign that we succeeded in bringing into play a datum not used before (the white knight). We were quite right to so regard it. In fact, to solve a problem is, essentially, to *find the connection between the data and the unknown.* Moreover we should, at least in well-stated problems, *use all the data,* connect each of them with the unknown. Thus, bringing one more datum into play is quite properly felt as progress, as a step forward.

In another example, we regarded as a favorable sign that an essential clause of the condition was appropriately taken into account by our formula. We were quite right to so regard it. In fact, we should *use the whole condition.* Thus, taking into account one more clause of the condition is justly felt as progress, as a move in the right direction.

In still another example, we regarded as a favorable sign the emergence of a simpler analogous problem. This also is justified. Indeed, analogy is one of the main sources of invention. If other means fail, we should try to *imagine an analogous problem.* Therefore, if such a problem emerges spontaneously, by its own accord, we naturally feel elated; we feel that we are approaching the solution.

After these examples, we can now easily grasp the general idea. There are certain mental operations typically useful in solving problems. (The most usual operations of this kind are listed in this book.) If such a typical operation succeeds (if one more datum is connected with the unknown—one more clause of the condition is taken into account—a simpler analogous problem is introduced) its success is felt as a sign of progress. Having understood this essential point, we can express with some

clearness the nature of still other signs of progress. All we have to do is to read down our list and look at the various questions and suggestions from our newly acquired point of view.

Thus, understanding clearly the nature of the unknown means progress. Clearly disposing the various data so that we can easily recall any one also means progress. Visualizing vividly the condition as a whole may mean an essential advance; and separating the condition into appropriate parts may be an important step forward. When we have found a figure that we can easily imagine, or a notation that we can easily retain, we can reasonably believe that we have made some progress. Recalling a *problem related to ours and solved before* may be a decisive move in the right direction.

And so on, and so forth. To each mental operation clearly conceived corresponds a certain sign clearly expressible. Our list, appropriately read, lists also signs of progress.

Now, the questions and suggestions of our list are simple, obvious, just plain common sense. This has been said repeatedly and the same can be said of the connected signs of progress we discuss here. To read such signs no occult science is needed, only a little common sense and, of course, a little experience.

4. *Less clearly expressible signs.* When we work intently, we feel keenly the pace of our progress: when it is rapid we are elated; when it is slow we are depressed. We feel such differences quite clearly without being able to point out any distinct sign. Moods, feelings, general aspects of the situation serve to indicate our progress. They are not easy to express. "It looks good to me," or "It is not so good," say the unsophisticated. More sophisticated people express themselves with some nuance: "This is a well-balanced plan," or "No, something is still

lacking and that spoils the harmony." Yet behind primitive or vague expressions there is an unmistakable feeling which we follow with confidence and which leads us frequently in the right direction. If such feeling is very strong and emerges suddenly, we speak of inspiration. People usually cannot doubt their inspirations and are sometimes fooled by them. In fact, we should treat guiding feelings and inspirations just as we treat the more clearly expressible signs of progress which we have considered before. Trust, but keep your eyes open.

Always follow your inspiration—with a grain of doubt.

[What is the nature of those guiding feelings? Is there some less vague meaning behind words of such aesthetic nuances as "well-balanced," or "harmonious"? These questions may be more speculative than practical, but the present context indicates answers which perhaps deserve to be stated: Since the more clearly expressible signs of progress are connected with the success or failure of certain rather definite mental operations, we may suspect that our less clearly expressible guiding feelings may be similarly connected with other, more obscure, mental activities—perhaps with activities whose nature is more "psychological" and less "logical."]

5. *How signs help.* I have a plan. I see pretty clearly where I should begin and which steps I should take first. Yet I do not quite see the lay-out of the road farther on; I am not quite certain that my plan will work; and, in any case, I have still a long way to go. Therefore, I start out cautiously in the direction indicated by my plan and keep a lookout for signs of progress. If the signs are rare or indistinct, I become more hesitant. And if for a long time they fail to appear altogether, I may lose courage, turn back, and try another road. On the other hand, if the signs become more frequent as I proceed, if they multiply, my hesitation fades, my spirits rise, and I move

with increasing confidence, just as Columbus and his companions did before sighting the island of San Salvador.

Signs may guide our acts. Their absence may warn us of a blind alley and save us time and useless exertion; their presence may cause us to concentrate our effort upon the right spot.

Yet signs may also be deceptive. I once abandoned a certain path for lack of signs, but a man who came after me and followed that path a little farther made an important discovery—to my great annoyance and long-lasting regret. He not only had more perseverance than I did but he also read correctly a certain sign which I had failed to notice. Again, I may follow a road cheerfully, encouraged by favorable signs, and run against an unsuspected and insurmountable obstacle.

Yes, signs may misguide us in any single case, but they guide us right in the majority of them. A hunter may misinterpret now and then the traces of his game but he must be right on the average, otherwise he could not make a living by hunting.

It takes experience to interpret the signs correctly. Some of Columbus's companions certainly knew by experience how the sea looks near the shore and so they were able to read the signs which suggested that they were approaching land. The expert knows by experience how the situation looks and feels when the solution is near and so he is able to read the signs which indicate that he is approaching it. The expert knows more signs than the inexperienced, and he knows them better; his main advantage may consist in such knowledge. An expert hunter notices traces of game and appraises even their freshness or staleness where the inexperienced one is unable to see anything.

The main advantage of the exceptionally talented may consist in a sort of extraordinary mental sensibility. With exquisite sensibility, he feels subtle signs of progress or notices their absence where the less talented are unable to perceive a difference.

[6. *Heuristic syllogism.* In section **2** we came across a mode of heuristic reasoning that deserves further consideration and a technical term. We begin by restating that reasoning in the following form:

> If we are approaching land, we often see birds.
> Now we see birds.
>
> ---
>
> Therefore, it becomes more credible that we are approaching land.

The two statements above the horizontal line may be called the *premises,* the statement under the line, the *conclusion.* And the whole pattern of reasoning may be termed a *heuristic syllogism.*

The premises are stated here in the same form as in section **2**, but the conclusion is more carefully worded. An essential circumstance is better emphasized. Columbus and his men conjectured from the beginning that they would eventually find land sailing westward; and they must have given some credence to this conjecture, otherwise they would not have started out at all. As they proceeded, they related every incident, major or minor, to their dominating question: "Are we approaching land?" Their confidence rose and fell as events occurred or failed to occur, and each man's beliefs fluctuated more or less differently according to his background and character. The whole dramatic tension of the voyage is due to such fluctuations of confidence.

The heuristic syllogism quoted exhibits a reasonable ground for a change in the level of confidence. To occasion such changes is the essential role of this kind of

reasoning and this point is better expressed by the wording given here than by the one in section **2**.

The general pattern suggested by our example can be exhibited thus:

> If A is true, then B is also true, as we know.
> Now, it turns out that B is true.
>
> ---
>
> Therefore, A becomes more credible.

Still shorter:

$$\frac{\begin{array}{c}\text{If } A \text{ then } B\\ B \text{ true}\end{array}}{A \text{ more credible}}$$

In this schematic statement the horizontal line stands for the word "therefore" and expresses the implication, the essential link between the premises and the conclusion.]

[7. *Nature of plausible reasoning.* In this little book we are discussing a philosophical question. We discuss it as practically and informally and as far from high-brow modes of expression as we can, but nevertheless our subject is philosophical. It is concerned with the nature of heuristic reasoning and, by extension, with a kind of reasoning which is nondemonstrative although important and which we shall call, for lack of a better term, *plausible* reasoning.

The signs that convince the inventor that his idea is good, the indications that guide us in our everyday affairs, the circumstantial evidence of the lawyer, the inductive evidence of the scientist, statistical evidence invoked in many and diverse subjects—all these kinds of evidence agree in two essential points. First, they do not have the certainty of a strict demonstration. Second, they are useful in acquiring essentially new knowledge, and even indispensable to any not purely mathematical or logical knowledge, to any knowledge concerned with the physical world. We could call the reasoning that underlies this kind of evidence "heuristic reasoning" or "inductive reasoning" or (if we wish to avoid stretching the meaning of existing terms) "plausible reasoning." We accept here the last term.

The heuristic syllogism introduced in the foregoing may be regarded as the simplest and most widespread pattern of plausible reasoning. It reminds us of a classical pattern of demonstrative reasoning, of the so-called "modus tollens of hypothetical syllogism." We exhibit here both patterns side by side:

Demonstrative	*Heuristic*
If A then B	If A then B
B false	B true
A false	A more credible

The comparison of these patterns may be instructive. It may grant us an insight, not easily obtainable elsewhere, into the nature of plausible (heuristic, inductive) reasoning.

Both patterns have the same first premise:

If A then B.

They differ in the second premise. The statements:

B false $\qquad\qquad$ B true

are exactly opposite to each other but they are of "similar logical nature," they are on the same "logical level." The great difference arises after the premises. The conclusions

A false $\qquad\qquad$ A more credible

are on different logical levels and their relations to their respective premises are of a different logical nature.

The conclusion of the demonstrative syllogism is of the

same logical nature as the premises. Moreover, this conclusion is fully expressed and is fully supported by the premises. If my neighbor and I agree to accept the premises, we cannot reasonably disagree about accepting also the conclusion, however different our tastes or other convictions may be.

The conclusion of the heuristic syllogism differs from the premises in its logical nature; it is more vague, not so sharp, less fully expressed. This conclusion is comparable to a force, has direction and magnitude. It pushes us in a certain direction: A becomes *more* credible. The conclusion also has a certain strength: A may become *much more* credible, or *just a little more* credible. The conclusion is not fully expressed and is not fully supported by the premises. *The direction is expressed and is implied by the premises, the magnitude is not.* For any reasonable person, the premises involve that A becomes more credible (certainly not less credible). Yet my neighbor and I can honestly disagree *how much* more credible A becomes, since our temperaments, our backgrounds, and our unstated reasons may be different.

In the demonstrative syllogism the premises constitute a *full basis* on which the conclusion rests. If both premises stand, the conclusion stands too. If we receive some new information that does not change our belief in the premises, it cannot change our belief in the conclusion.

In the heuristic syllogism the premises constitute only one *part of the basis* on which the conclusion rests, the fully expressed, the "visible" part of the basis; there is an unexpressed, invisible part, formed by something else, by inarticulate feelings perhaps, or by unstated reasons. In fact, it can happen that we receive some new information that leaves our belief in both premises completely intact, but influences the trust we put in A in a way just opposite to that expressed in the conclusion. To find A more plausible on the ground of the premises of our heuristic syllogism is only reasonable. Yet tomorrow I may find grounds, not interfering at all with these premises, that make A appear less plausible, or even definitively refute it. The conclusion may be shaken and even overturned completely by commotions in the invisible parts of its foundation, although the premises, the visible part, stand quite firm.

These remarks seem to make somewhat more understandable the nature of heuristic, inductive, and other sorts of not demonstrative plausible reasoning, which appear so baffling and elusive when approached from the point of view of purely demonstrative logic. Many more concrete examples, the consideration of other kinds of heuristic syllogism, and an investigation of the concept of probability and other allied concepts seem to be necessary to complete the approach here chosen; *cf.* the author's *Mathematics and Plausible Reasoning*.]

Heuristic reasons are important although they prove nothing. To clarify our heuristic reasons is also important although behind any reason clarified there are many others that remain obscure and are perhaps still more important.

Specialization is passing from the consideration of a given set of objects to that of a smaller set, or of just one object, contained in the given set. Specialization is often useful in the solution of problems.

1. *Example.* In a triangle, let r be the radius of the inscribed circle, R the radius of the circumscribed circle, and H the longest altitude. Then

$$r + R \leqq H.$$

The proposed theorem is of an unusual sort. We can scarcely remember any theorem about triangles with a similar conclusion. If nothing else occurs to us, we may test some *special case* of this unfamiliar assertion. Now, the best known special triangle is the equilateral triangle for which

$$r = \frac{H}{3} \qquad R = \frac{2H}{3}$$

so that, in this case, the assertion is correct.

If no other idea presents itself, we may test the *more extended special case* of isosceles triangles. The form of an isosceles triangle varies with the angle at the vertex and there are two extreme (or limiting) cases, the one in which the angle at the vertex becomes 0°, and the other in which it becomes 180°. In the first extreme case the base of the isosceles triangle vanishes and visibly

$$r = 0 \qquad R = \frac{1}{2}H$$

thus, the assertion is verified. In the second limiting case, however, all three heights vanish and

$$r = 0 \qquad R = \infty \qquad H = 0.$$

The assertion is not verified. We have proved that the proposed theorem is false, and so we have solved our problem.

By the way, it is clear that the assertion is also false for very flat isosceles triangles whose angle at the vertex is nearly 180° so that we may "officially" disregard the extreme cases whose consideration may appear as not quite "orthodox."

2. "L'exception confirme la règle." "The exception