

# Why is climate sensitivity so unpredictable?

## Supplementary material

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### Feedback terminology

In this section we derive the parameters we use in discussing feedbacks. We employ the standard terminology of feedback analysis from electronics (e.g., *S1-S5*). It is, however, different from the terminology found in much of the climate literature, which reverses the definition of a feedback factor and a gain (e.g., *S6*).

Consider the net radiation balance at the top of the atmosphere,  $R$ , to be the sum of the outgoing longwave radiation,  $F$ , and the net absorbed solar radiation,  $S$  (all fluxes are defined as positive downwards). If the climate is in equilibrium then  $R = 0$ , so  $F = -S$ , and the global mean temperature has its equilibrium value. Now let  $\Delta R_f$  be some specified, constant, radiative forcing (here, due to anthropogenic emissions of  $\text{CO}_2$ ).  $\Delta R_f$  is assumed small compared with  $F$  and  $S$  so that the changes it introduces in system dynamics are all small and linear in the forcing. In order to attain a new equilibrium state, the radiative balance must change by an amount  $\Delta R$  that is equal and opposite to the forcing. That is,  $\Delta R = -\Delta R_f$ . In other words, in the new equilibrium state the longwave and shortwave radiation must adjust:  $\Delta R = \Delta F + \Delta S$ , and the temperature reaches a new equilibrium value, displaced from the old by an amount

$\Delta T$ . The new equilibrium temperature,  $\Delta T$ , can be found assuming a first-order Taylor-series expansion:

$$\Delta R \approx \frac{dR}{dT} \Delta T. \quad (\text{S1})$$

Feedbacks are only meaningfully defined in the context of a specified reference system, against which the effects of incorporating the feedbacks can be evaluated. A standard reference case presented in textbooks is a blackbody planet, in which radiative forcing is balanced only by adjustments in longwave flux via the Stefan-Boltzmann relationship,  $F = -\sigma T^4$  (S5). We can write  $\Delta T_e = \lambda_0 \Delta R_f$ . From Eq. S1,  $\lambda_0 = 1/(4\sigma T_e^3)$ , where  $T_e$  is the planetary blackbody temperature of 255 K (S4) giving  $\lambda_0 = 0.26 \text{ K}/(\text{W m}^{-2})$ .

When other climate fields are allowed to adjust in response to a change in forcing, their impacts on the forcing (i.e., feedbacks) must be accounted for. We first consider the linear case, in which these changes are small and proportional to  $\Delta T$ . In this case, if  $\alpha_i$  is the  $i^{\text{th}}$  climate field,

$$\Delta T = \lambda_0 \Delta R_f + \sum_i f_i \Delta T. \quad (\text{S2})$$

where

$$f_i \equiv \lambda_0 \left\{ \left( \frac{\partial R}{\partial \alpha_i} \right)_{\alpha_{j,j \neq i}} \frac{d\alpha_i}{dT} \right\}. \quad (\text{S3})$$

$f_i$  is the nondimensional *feedback factor* for the  $i^{\text{th}}$  feedback process. It can be seen that  $f_i$  is proportional to the fraction of the response that is fed back into the forcing. Defining  $f \equiv \sum_i f_i$ , gathering up the  $\Delta T$ s and rewriting:

$$\Delta T = \frac{\lambda_0}{(1-f)} \Delta R_f. \quad (\text{S4})$$

The *gain* of the system,  $G$ , is the proportion by which the system response has changed relative to the reference case, as a result of including the feedbacks.

$$G \equiv \frac{\Delta T}{\Delta T_0} = \frac{1}{(1-f)}. \quad (\text{S5})$$

Thus if  $f < 0$ , the net feedbacks are negative,  $G < 1$ , and the system response is damped. If  $0 < f < 1$ , the net feedbacks are positive,  $G > 1$ , and the system response is amplified. If  $f \geq 1$ , runaway growth ensues, no new equilibrium can be established via the assumed physics, and the gain is undefined.

## The analogy with a Bayesian framework

In the Bayesian framework a prior assumption about the probability distribution of climate sensitivity is modified by a comparison with data (observations or models), leading to a modified, posterior, distribution (*S7*, *S8*). In these terms, an analysis can be written

$$h_{post}(\Delta T) = \frac{p(f_{obs}|\Delta T)}{p(f)} h_{prior}(\Delta T), \quad (S6)$$

where  $h_{post}(\Delta T)$  is the result of the analysis - the posterior probability distribution for climate sensitivity.  $p(f_{obs}|\Delta T)/p(f)$  is the normalized probability that a observation yielding  $f_{obs}$  is consistent with a climate sensitivity  $\Delta T$ . Statistical studies suggest that a Gaussian distribution in  $p(f_{obs}|\Delta T)$  is appropriate (*S7*), similar to our choice for  $h(f)$ . In general, a prior assumption can be made about the distribution of any climate variable,  $z$ , in which case a conversion is needed:  $h_{prior}(z)dz = h_{prior}(\Delta T)dT$ . A natural choice is that all feedbacks are equally likely ( $z = f$ ,  $h_{prior}(f)$  is a constant). Since  $f = 1 - \Delta T_0/\Delta T$ ,  $h_{prior}(\Delta T) \sim \Delta T_0/(\Delta T)^2$ . This, together with a Gaussian for  $p(f_{obs}|\Delta T)$ , makes Eq. ?? algebraically equivalent to Eq. 3 in our main text. We note that in our analysis starting from Eq. 1 in the main text, the factor  $1/(\Delta T)^2$  arises from the mathematical relationship between  $\Delta T$  and  $f$ , rather than from a prior assumption.

Some studies have also considered a uniform prior distribution on climate sensitivity (*S9*, *S10*). In which case  $z = \Delta T$ ,  $h_{prior}(\Delta T)$  is a constant, and the factor of  $1/(\Delta T)^2$  is omitted in  $h_{post}(\Delta T)$ . For a given  $p(f_{obs}|\Delta T)$  this leads to a fatter tail to the distribution.

## Nonlinear feedbacks

In Eq. S5 the feedbacks are independent of the system response,  $\Delta T$ . Here we briefly consider the effects of adding plausible nonlinear feedbacks. Taking the Taylor series in Eq. S1 to second order gives a quadratic relationship between  $\Delta T$  and  $\Delta R$ :

$$\Delta R \approx R' \Delta T + \frac{1}{2} R'' \Delta T^2, \quad (\text{S7})$$

where we have used the shorthand  $()' \equiv d/dT()$ . To estimate the new  $\Delta T$  we approximate the solution to Eq. S8; in analogy with Eq. S5 we now have

$$\Delta T \approx \frac{\lambda_0 \Delta R_f}{1 - f - \frac{1}{2} \lambda_0 R'' \Delta T_0}, \quad (\text{S8})$$

where we have replaced the term in  $\Delta T^2$  by  $\Delta T \Delta T_0$  to first order, and again used the fact that  $\Delta R = -\Delta R_f$ . The first nonlinearity we consider is that at higher temperatures the  $T^4$  dependence of the Stefan-Boltzmann equation means the climate system is able to more effectively compensate for radiation perturbations than at lower temperatures. The second nonlinearity is that the water vapor feedback depends on the moisture content of the air, which via the Clausius-Clapeyron relation (S5) is a nonlinear function of temperature. Physically, the Stefan-Boltzmann feedback becomes more negative and the water vapor feedback becomes less positive as the temperature increases. Both effects drive the system towards greater stability. Since now we are considering only the effects of the Stefan-Boltzmann and water vapor feedback irregularities so that  $R' = F'$ ,  $R'' = F''$ .

We write  $F = -\sigma(T - \Theta(T))^4$ , where  $\Theta(T)$  reflects the effect of water vapor, and  $T$  represents the surface temperature. At equilibrium in the unperturbed case,  $T_0 = 288$  K, and  $T_0 - \Theta(T_0) = 255$  K, the planetary blackbody temperature. Now,

$$F' = -4\sigma(T - \Theta)^3(1 - \Theta') \equiv -\frac{1}{\lambda_0}(1 - \Theta'), \quad (\text{S9})$$

so that

$$F'|_{T_0} = \lambda_0^{-1}(f_{wv}(T_0) - 1), \quad (\text{S10})$$

where  $f_{wv}(T) = \Theta'(T)$  is the water vapor feedback. At the unperturbed equilibrium temperature;  $f_{wv}(T_0) \approx 0.4$  as estimated from climate models (S11, S12).

We suppose that the effect of water vapor is dependent on the logarithm of the saturated atmospheric moisture content, reflecting the fact that as temperature, moisture, and infrared opacity of the atmosphere increase, the water vapor *feedback* becomes progressively less effective. So let  $\Theta \propto \ln[e_{sat}(T)]$ , where  $e_{sat}(T)$  is the saturation water vapor given by the Clausius-Clapeyron equation (S5). In that case

$$f_{wv}(T) = \Theta' \propto \frac{d}{dT} \ln(e_{sat}(T)) \propto \frac{1}{T^2}. \quad (\text{S11})$$

Hence  $f'_{wv}(T) = -2f_{wv}(T)/T$ .

Differentiating Eq. S10 gives:

$$F'' = \left(\frac{1}{\lambda_0^2}\right)\lambda_0'(1 - f_{wv}) + \frac{1}{\lambda_0}f'_{wv}. \quad (\text{S12})$$

Since both  $\lambda_0'$  and  $f'_{wv}$  are negative, we see that  $F'' < 0$ .

Evaluating Eq. S12 at  $T = T_0$  yields

$$F''|_{T_0} = \frac{3(f_{wv}(T_0) - 1) + f'_{wv}(T_0)(T_0 - \Theta(T_0))}{\lambda_0(T_0 - \Theta(T_0))}. \quad (\text{S13})$$

Then  $\Delta T$  can be found from Eq. S8.

The effect of including the quadratic terms is to incorporate an extra negative feedback factor. For characteristic values of  $f_{wv}(T_0) = 0.4$  (S11, S12) and  $T_0 - \Theta(T_0) = 255$  K, the effective extra negative feedback factor is about  $3 \times 10^{-3}$  per degree of climate change. For  $\Delta T \sim 3$  °C,  $f$  changes by  $\sim 0.01$ . In other words, for the physics analyzed here the effect of these extra terms is quite small.

**Is it possible to remove the skewness?** One can address the question of whether the skewness in the climate sensitivity curve towards larger climate changes (i.e., Figures 3 and 4) is unavoidable, or whether, if the feedbacks were sufficiently nonlinear, the tail might be eliminated.

Solving the full quadratic Eq. S8, we find that to second order in the general case, for total feedback  $f$ ,

$$\Delta T = \frac{\Delta T_0}{1-f} \left( 1 - \frac{1}{2} \left( \frac{\Delta T_0}{1-f} \right) f' \right), \quad (\text{S14})$$

where  $\Delta T_0 \equiv \lambda_0 \Delta R_f$ ,  $f \equiv \lambda_0 R' + 1$ , and  $f' \equiv \lambda_0 R''$ .

The condition for removing the skewness of the climate sensitivity curve is that there be no curvature in  $\Delta T$  as  $f$  varies. This requires  $f' = 2f(1-f)/\Delta T_0$ . Therefore, for a typical estimated total feedback of  $f \sim 0.65$ , and  $\Delta T_0 \sim 3^\circ\text{C}$ , removing the skewness would require that  $f$  diminish by about 0.15 per degree of climate change, or in other words, significantly more than we found to be reasonable in the above section.

### **Constraining $\Delta T_{max}$**

To some extent, the high  $\Delta T$  tail in the distribution  $h_T(\Delta T)$  reflects the fact that we have assumed a Gaussian distribution for the underlying feedbacks, leading to a finite probability of values of  $f$  close to one. To investigate the effect of this, we have recomputed  $h_T(\Delta T)$  assuming  $h_f(f)$  is the Gaussian multiplied by a Heaviside function to eliminate such values of  $f$  producing values of  $\Delta T$  greater than some assumed maximum  $\Delta T_{max}$ . This cutoff has little impact on the resulting  $h_T(\Delta T)$ . An example for  $\Delta T_{max} = 8^\circ\text{C}$  is shown in Figure S1. The cumulative probabilities for the  $\Delta T \geq 5^\circ\text{C}$  then are essentially independent of  $\sigma_f$ . Therefore attempts to constrain  $\sigma_f$  become even less important for constraining the probabilities of large  $\Delta T$ .

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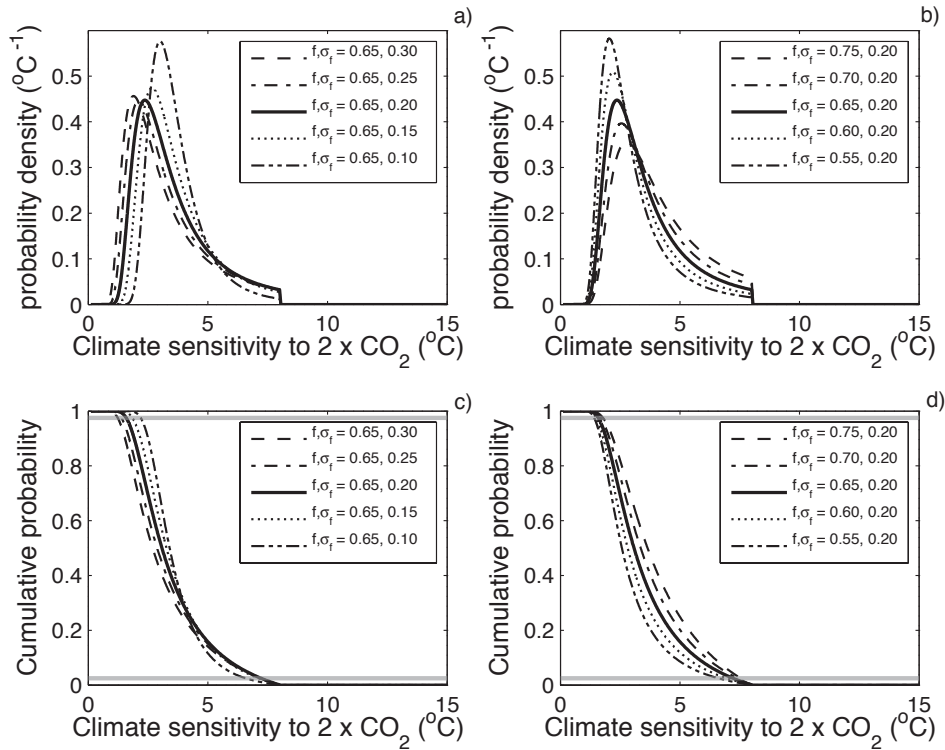


Figure 1: Panels (a) and (b): Probability distributions  $h_T(\Delta T)$  for a range of values of  $\bar{f}$  and  $\sigma_f$  in the case where the underlying probability distribution for the feedbacks,  $h(f)$ , is a truncated Gaussian, set to zero for all values of  $f$  that would produce  $\Delta T > \Delta T_{max} = 8^\circ\text{C}$  and then renormalized. Panels (c) and (d): the corresponding cumulative distributions  $p_{cum}(\Delta T)$ , corresponding to the truncated Gaussian. The effect of the cutoff is to force  $p_{cum}(\Delta T)$  to vanish for  $\Delta T > \Delta T_{max}$  for all  $\sigma_f$  and to become almost independent of  $\sigma_f$  for  $\Delta T \geq \approx 5^\circ\text{C}$ . That is: in eliminating the tail of the distribution we remove the dependence of moderately large sensitivities on  $\sigma_f$ .