ESS 314 "Math Camp", Fall 2013, to be handed in on Friday, October 4th in class.

The following problems draw heavily from the math reading posted on the web site. You should have read those notes, and they will be helpful here. Written (and neat!) solutions to all problems should be handed in at the next lab session.

I want to see neat versions of your answers, and a linear sequence of reasoning. Work them out on a scrap of paper first if needs be, and then write them down on the paper you turn in. I don't want to see multiple crossings-out or hard-to-follow working that jumps around on the page. Read the guidance handout from the start of the class, and follow the style suggested. Sloppy and untidy work will be penalized heavily, even if the answer is right.

There are scanned examples of solutions to similar problems on the class web site that show the standard of written work we expect. These worked solutions are also helpful for doing the problems below.

1. **Estimation question.**

How long can 25 people last in a room the size of JHN026, before oxygen levels fall below safe values? Give your answer to two significant figures.

This is an estimation question – you are going to have to make some assumptions about how much oxygen there is in the room, the rate that the average person consumes it, and what concentration becomes dangerous.

You are just trying to make a crude estimate here. Just make some sensible assumptions (make sure to list them), and you can probably do a little web research to find out the necessary parameters.

2. **Exponential algebra.**

Humanoids cannot live permanently in an atmosphere with a density less than 60% of that at Earth's sea level. Assuming the scale height of Earth's atmosphere is 10 km, what is the highest altitude one should expect permanent habitation? (Hint: Look at the math notes for guidance on how to do this).

3. **Derivatives.**

Compute derivatives of the following functions:

a) $f(\theta) = \tan\theta = \frac{\sin\theta}{\cos\theta}$

b) $y(x) = x^{e^x}$

c) $g(t) = \sqrt{\frac{t^4-1}{1-e^t}}$

Give your answer in explicit form (i.e. dy/dx =…)

Hint for part b) – take logs of both sides.
Hint for part c) – one way to do this is to operate on both sides to get a simpler expression on the right hand side, and then use the quotient rule.

4. **Maxima and Minima**.
Find the values of x at which the following function is maximum and minimum

$$f(x) = \frac{x^3}{3} + \frac{x^2}{2} - 2x + 1$$

5. **Calculus of two variables, surfaces and slopes.**

If $h(x, y) = (x-2)^2 + (y-4)^2$

(i) Sketch the shape of the surface.
[hint: this is a 3-D surface – just make a contour plot, or plot cross-sections through the shape, but pay attention to the contour separation]
(ii) Give the general expression for the gradient ∇h in terms of x and y.
(iii) Calculate the magnitude of the steepest slope at (x, y) = (0, 6) and (2, 4).

6. **Taylor series**

Let $f(x) = x^{\frac{1}{3}}$. Using a first-order Taylor series show *(without using a calculator)* that

$$f(27.8) \approx 3 + \frac{8}{270}$$

Now using a calculator, find out to how many significant figures the approximate expression is accurate.

7. **Forces, acceleration, velocities, and distances**.
An interplanetary expedition decides to make a stop at a rocky asteroid of 1 km radius. You leave the spacecraft to walk around on the surface. In your enthusiasm for being outside the spacecraft you start to jump up and down. Suddenly, to your horror, you realize that you are not coming back to the asteroid, but are instead heading for outer space. You have exceeded the “escape velocity” of the asteroid and you will have just enough air in your space suit to contemplate:

(i) The minimum vertical speed, u_i, you were going when you left the surface. See hints, below.

(ii) For the same initial velocity, show that the distance you could have jumped on the surface of Earth if you had survived to complete the journey is given by the expression:

$$\Delta R \approx \frac{u_i^2}{2} \frac{R_E^2}{GM},$$

where R_E is the Earth's radius, and M is its mass.

(iii) What is the value of ΔR and could have become a professional basketball player?

Hints: This is an involved question. Take it step-by-step.

I. The trick here is that you want to find your initial speed such that your final speed is zero at r = ∞ (i.e., you just escaped). You need to integrate the change in acceleration as you move from r = 1 km to r = ∞.

II. The gravitational acceleration outside a body of mass M is:

$$a = -\frac{GM}{r^2}$$

where the Gravitational Constant is 6.673×10^{-11} m^3/kg/s^2. The product rule implies

$$a = \frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = \frac{dv}{dr}v$$

which can be used to convert the acceleration equation into an equation that is easily integrated.

III. Remember mass equals density times volume. Assume the density of the asteroid is $\rho = 3{,}000$ kg/m^3, and that its volume = 4/3 π r^3 (i.e., that it is spherical).

8. At time t = 0, a wave has the following shape

$$h(x,t=0) = \mathrm{Re}\left(h_0 e^{ikx}\right) = h_0 \cos(kx),$$

where $k = 2\pi/\lambda$, and λ is the wavelength (let λ = 1 m). Re() means take the real part of the function.

a) First, sketch a graph of *h(x,t=0)*, and find the location, x_0, of the first trough that lies to the right of $x = 0$.

The wave propagates to the right, with a function given by

$$h(x,t) = \mathrm{Re}\left(h_0 e^{i(kx-\omega t)}\right) = h_0 \cos(kx - \omega t)$$

where $\omega = 2\pi f$, and f is the frequency (let $f = 1$ Hz = 1 s^{-1}).

b) What is the speed of the wave? Give an expression for that speed in terms of ω and k.

c) Draw a graph of x as a function of t, showing how the location of the trough that was at x_0 at t=0 changes with time. Find the location of the trough that was at x_0 after 4s have elapsed.

9. If z = x + iy, write the following in "rectangular" (a+ib) form

$$\frac{1+z}{1-z}$$

Hint: turn the denominator into a real number, using the fact that an complex number multiplied by its conjugate always creates a real number.

10. Calculate $\sqrt{i}$ and write the answer as a complex number. *Hints*: Write i as an imaginary exponential, take the square root of this exponential and express the result as a complex number.