Sample answers showing the expected standard

Qu 3 lookalike

i/ compute derivative of $\tan^{-1}\theta = \dfrac{\cos\theta}{\sin\theta}$

Use quotient rule: $\dfrac{d}{d\theta}\tan^{-1}\theta = \dfrac{d}{d\theta}\left(\dfrac{\cos\theta}{\sin\theta}\right)$

$$= \frac{\sin\theta . -\sin\theta - \cos\theta \cdot \cos\theta}{\sin^2\theta}$$

$$= \frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta}$$

$$= \underline{\underline{\frac{-1}{\sin^2\theta}}} \qquad \text{since } \sin^2\theta + \cos^2\theta = 1$$

ii/ compute derivative of $q(z) = z^{\cos z}$

take $\log_e$ of both sides:

$$\ln q = \ln\left(z^{\cos z}\right) \cancel{= \cos z}$$

$$= \cos z \cdot \ln z$$

differentiate: $\dfrac{1}{q}\dfrac{dq}{dz} = \dfrac{\cos z}{z} + \ln z \times -\sin z$ (chain rule)

Qu3 ii cont^d

$$\Rightarrow \frac{dq}{dz} = q\left(\frac{\cos z}{z} - \sin z \cdot \ln z\right)$$

$$= z^{\cos z}\left(\frac{\cos z}{z} - \sin z \ln z\right)$$

iii compute derivative of $g(t) = \left(\dfrac{t^2-1}{1-\sin t}\right)^{1/3}$

cube both sides to simplify $g^3 = \dfrac{t^2-1}{1-\sin t}$

differentiate (quotient rule) $3g^2\dfrac{dg}{dt} = \dfrac{(1-\sin t)\cdot 2t - (t^2-1)\cdot -\cos t}{(1-\sin t)^2}$

$$\Rightarrow \frac{dg}{dt} = \frac{1}{3g^2}\left\{\frac{2t(1-\sin t) + \cos t(t^2-1)}{(1-\sin t)^2}\right\}$$

$$= \frac{1}{3}\left(\frac{1-\sin t}{t^2-1}\right)^{2/3}\left\{\frac{2t(1-\sin t)+\cos t(t^2-1)}{(1-\sin t)^2}\right\}$$

$$= \frac{1}{3}\frac{\left[2t(1-\sin t) + \cos t(t^2-1)\right]}{(t^2-1)^{2/3}(1-\sin t)^{4/3}}$$

Qu 4 (lookalike)

find values of x at which the following function is a maximum and minimum

$$f(x) = x^3 + 2x^2 + x + 1$$

find x where $df/dx = 0$

solve $\quad 3x^2 + 4x + 1 = 0$

$$\Rightarrow \quad (3x+1)(x+1) = 0$$

$x = -1/3,\ x = -1$ are sol^n for $f_x = 0$

which is min/max?

$$\frac{d^2f}{dx^2} = 6x + 4$$

@ $x = -1/3$, $f_x = 0$, $f_{xx} = 2 \Rightarrow$ MINIMUM

@ $x = -1$, $f_x = 0$, $f_{xx} = -2 \Rightarrow$ MAXIMUM

ide working:

$x = -1$

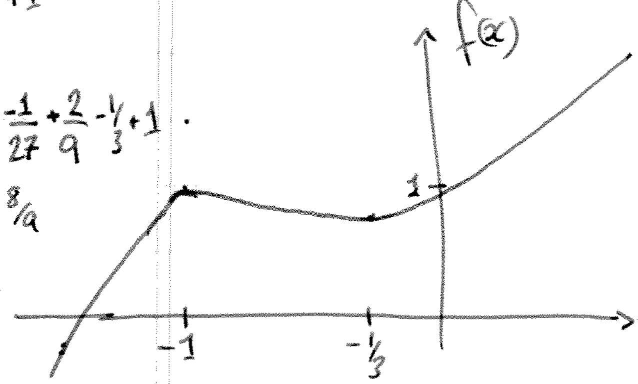
$f(x) = -1 + 2 - 1 + 1$

$= +1$

$x = -1/3$

$f(x) = -\frac{1}{27} + \frac{2}{9} - \frac{1}{3} + 1$

$= 8/9$



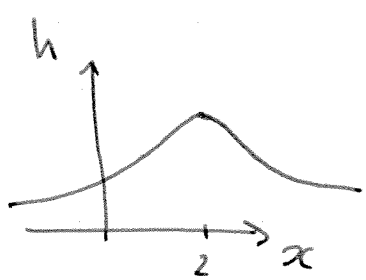
sketch of curve

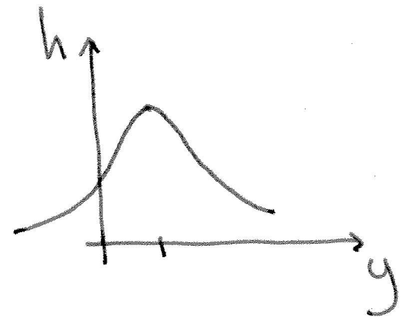
Qu5
lookalike

$$h(x,y) = e^{-(x-2)^2} + e^{-(y-1)^2}$$

i, sketch the shape of the surface

- surface is bell shaped in x & y dir's, and is centred on $(x,y) = (2,1)$





ii, Give general expression for ∇h

$$\nabla h = \underset{\sim}{i}\frac{dh}{dx} + \underset{\sim}{j}\frac{dh}{dy}$$

$$\nabla h = \underset{\sim}{i} \, . -2(x-2)e^{-(x-2)^2} + \underset{\sim}{j} \, -(y-1)e^{-(y-1)^2}$$

iii, Magnitude of steepest slope at $(0,6)$, $(2,4)$

$$|\nabla h| = 4(x-2)^2 e^{-2(x-2)^2} + (y-1)^2 e^{-2(y-1)^2}$$

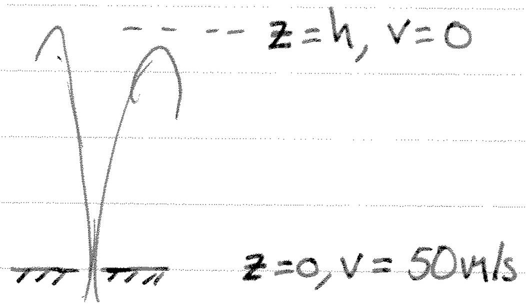
Using calculator $|\nabla h| @ (0,6) = 5.4 \times 10^{-3}$

$|\nabla h| @ (2,4) = 1.4 \times 10^{-7}$

Qu 6 lookalike

A geyser erupts from the ground with speed of $50\,ms^{-1}$. Assuming the only force acting is gravity, how high will the top of the geyser reach?

Sketch



at top of geyser, velocity will be 0.

let a = acceleration of fluid parcels $= -g$ $\quad g = 9.8\,ms^{-1}$

TRICK!

$$a = \frac{dv}{dt} = \frac{dv}{dz} \cdot \frac{dz}{dt}$$

but $\frac{dz}{dt} = v \Rightarrow a = v\frac{dv}{dz}$ (a lot easier to integrate)

so $$v\frac{dv}{dz} = -g$$

solve o.d.e.

$$\Rightarrow v\,dv = -g\,dz$$

QuG cont^d

integrate between ground and top of fountain:

$$\int_{v=50}^{v=0} v\,dv = -g\int_{z=0}^{z=h} dz$$

$$\left.\frac{v^2}{2}\right|_{50}^{0} = -gz\Big|_{0}^{h}$$

$$\Rightarrow -\frac{50^2}{2} = -gh$$

$$\Rightarrow h = \frac{50^2}{2 \cdot g} \sim \frac{250\cancel{0}}{2 \times 1\cancel{0}}$$

[Units $\frac{(ms^{-1})^2}{ms^{-2}} = m$ ✓]

$$\underline{\underline{h = 125\ m}}$$

(~300ft, answer seems reasonable)

Qu 8 lookalike

$$f(z) = \frac{1-z}{1+z} \quad \text{where } z = x+iy$$

write in polar and rectangular form

$$f(z) = \frac{1-(x+iy)}{1+(x+iy)}$$

$$= \frac{(1-x) - iy}{(1+x) + iy}$$

make denominator real, by multiplying by conjugate

$$f(z) = \frac{(1-x)-iy}{(1+x)+iy} \times \frac{(1+x)-iy}{(1+x)-iy}$$

$$= \frac{1-x^2-y^2 + i\left(y(1-x) - y(1+x)\right)}{(1+x)^2 + y^2}$$

$$= \frac{1-x^2-y^2}{(1+x)^2+y^2} + i\,\frac{-2xy}{(1+x)^2+y^2}$$

(a+ib form)

Polar form:- if $a+ib = re^{i\theta}$

then $r^2 = a^2+b^2$; $\theta = \tan^{-1} b/a$

8 cont^{d}

from previous $\quad a = \dfrac{1-x^2-y^2}{(1+x)^2+y^2}$, $b = \dfrac{-2xy}{(1+x)^2+y^2}$

$$r^2 = a^2 + b^2 \qquad \theta = \tan^{-1} b/a$$

This is probably the most compact way to write the answer – everything else is going to be messy.

Qu 9 lookalike

Calculate $(-i)^{1/3}$

Use exponential notation $\quad -i = 0 + i \times -1$

from above $\quad a = 0, b = -1 \Rightarrow r = 1 \quad \theta = \tan^{-1}(-1) = -\pi/2$

So $\quad (-i)^{1/3} = \left(e^{-i\pi/2}\right)^{1/3} = e^{-i\pi/6}$

$$\Rightarrow (-i)^{1/3} = \cos \pi/6 - i \sin \pi/6$$

$$= \cos 30^\circ - i \sin 30^\circ$$

$$= \frac{\sqrt{3}}{2} - i/2$$

$\sin 30 = 1/2$

$\cos 30 = \frac{\sqrt{3}}{2}$