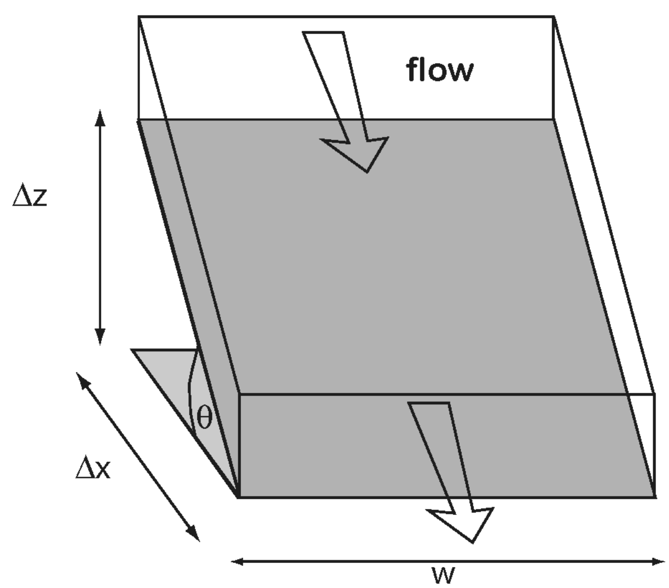
Erosion in a river channel – John Booker and Gerard Roe

In this section of the notes, we derive an equation for the physics that controls the shape of river profiles in mountainous regions. It works very well, and the lab exercise will hopefully convince you of that.

Let's being by proposing something reasonable. In order to perform erosion, energy must be used. Rivers flow downhill and in the process convert potential energy of the water into kinetic energy (i.e., motion), work (i.e., distance done against resistance), and ultimately, of course, heat (viscous dissipation of other forms of energy due to turbulence and friction).

A reasonable initial assumption is that the rate of erosion in a channel bed is proportional to the rate that a river converts potential energy into other forms of energy, divided by the area of the channel bed. The thinking goes that firstly, the greater the rate at which potential energy is converted, the more is available to erode the channel bed. And that secondly, the greater the area of the channel bed, the larger the area over which that energy is dissipated and so the lower the erosion.

$$\dot{e} = \frac{\text{rate potential energy is lost}}{\text{area of channel bed}}$$



Referring to the above sketch, if a slab of water of mass, m, that falls down a distance, Δz, then

Potential energy loss = $mg\Delta z$.

The *rate* of potential energy loss at any given point on the channel is then given by the *rate* that mass flows mass across a piece of the channel of length Δx. Q, the discharge is exactly this: $Q = dm/dt$. Q therefore has units of [kg s^{-1}]. Therefore

Rate potential energy is lost = -QgΔz

Next, the area of the segment of the channel bed we are considering is wΔx. Therefore we can say that

$$\dot{e} \propto \frac{\text{rate potential energy is lost}}{\text{area of channel bed}} = \frac{-Qg\Delta z}{w\Delta x}.$$

Finally then, the slope of the channel is $\Delta z/\Delta x \approx dz/dx$. And so the erosion rate can be written as

$$\dot{e} \propto -g(\frac{Q}{w})\frac{dz}{dx} = -K(\frac{Q}{w})\frac{dz}{dx},$$

where K is a constant, known as the erosivity, which is related to the rock strength and the erodibility of the rock. This is called the *stream power erosion law* (for obvious reasons). There are several other formulations, which are a subject of very active research (including in this department). The particular version derived above captures some very fundamental properties of erosion in a channel bead: a) erosion increases if discharge increases (which makes sense); b) erosion increases if channel slope increases (which makes sense); the erosion rate decreases if the channel width increases (which makes sense); and finally the erosion rate increases more erodible the rock (which makes sense). Next, we use this erosion law to predict the shape of a river channel.

The longitudinal profile of a river

The stream power erosion law can be used to predict how a river channel changes elevation with distance from the drainage divide in an actively uplifting mountain range. We make several further assumptions in addition to those already made (which will be discussed in class). The tectonic forcing creating the tendency for mountain uplift is called the rock uplift rate, *U*. We assume that this is uniform across the mountain range. Next we assume that the mountain belt is in steady state. That is, there is a balance between the rock uplift rate and the erosion rate, such that the channel is neither moving up or down over time. Lastly we assume that the rock erosivity is uniform in space.

We get to work on the stream power law a bit first, using some of the observed relationships between width, drainage area and precipitation.

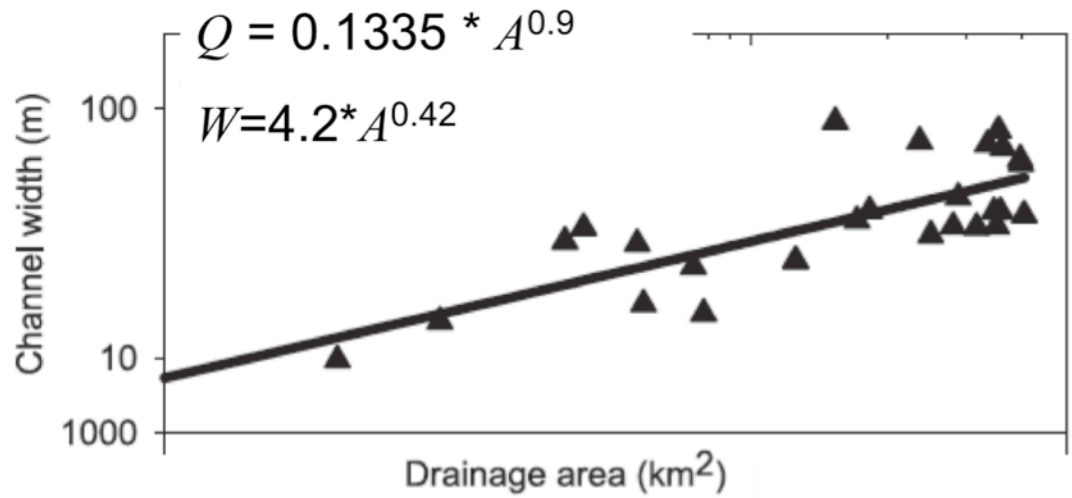


Figure of channel width vs. downstream distance for the Clearwater river, Olympic Mountains, WA, USA. It shows that, to a fair approximation, the channel width varies as the square root of the typical discharge (or, equivalently, the drainage areas) From Tomkin et al. (2003)

Firstly, from the above data set (and many others) we find that

$$w = kQ^{\frac{1}{2}},$$

which means that the stream power law can be written as

$$\dot{e} = -K'Q^{\frac{1}{2}}\frac{dz}{dx},$$

where K’ is a constant. Next, assuming that the precipitation in uniform the discharge is just equal to the precipitation rate multiplied by the upstream drainage area. Or in other words, Q=PA. Hence

$$\dot{e} = -K'(PA)^{\frac{1}{2}}\frac{dz}{dx}.$$

Finally we can relate the drainage area to the position on the channel: $A = k_a x^2$. So substituting this into the above equation gives

$$\dot{e} = -K''P^{\frac{1}{2}}x\frac{dz}{dx},$$

where *K’’* is a constant. Okay, now we have all the pieces.

In an active mountain belt that is maintaining a constant altitude, we must have a balance between the erosion rate and the rock uplift rate, $U = \dot{e}$, and so

$$U = -K'' P^{\frac{1}{2}} x \frac{dz}{dx}.$$

This is a first-order differential equation for z as a function of x. The equation can be solved if some boundary conditions are stipulated. Let L be the length of the channel. At x = L the channel is at the elevation of the floodplain. We can all that z = 0.

We can rearrange the equation above to have the dz's and dx's on different sides of the equation

$$-\frac{U}{K'' P^{1/2}} \frac{dx}{x} = dz$$

By assumption U, K, and P are all constant, so we can write

$$-\left(\frac{U}{KP^{1/2}}\right) \int \frac{dx}{x} = \int dz,$$

dropping the primes on the K. This has a general solution

$$z = -\left(\frac{U}{KP^{\frac{1}{2}}}\right) \ln(x) + C,$$

where C is an integration constant. Now this is where the boundary condition comes in. At $x=L$, $z=0$. Substituting this into the above equation gives $C = (U/(KP^{1/2}))\ln(L)$. Putting this value for C into the above equation gives

$$z = \left(\frac{U}{KP^{\frac{1}{2}}}\right) \left[\ln(L) - \ln(x)\right] = \left(\frac{U}{KP^{\frac{1}{2}}}\right) \ln\left(\frac{L}{x}\right).$$

This is our prediction for the elevation of the river channel as a function of distance. The theory works well for many channels, as we will see in the lab this week. However something strange happens near the drainage divide (at $x = 0$). As $x \to 0$, the theory predicts that $z \to \infty$. Obviously this does not happen. In practice, at some point near the drainage divide, there is not enough discharge for a real river or stream channel to form. Instead erosion happened on hillslopes instead, via landslides and debris flows.