# ESS314

# Heat, Conduction, and Diffusion by John Booker and Gerard Roe

# Introduction

There are many interesting questions that involve the physical transfer of heat. These include:

- 1. How does a bathylith cool?
- 2. Why does the temperature of a cave remain relatively constant year round?
- 3. Why are there hot springs unrelated to volcanism?
- 4. How could we directly infer the temperature change at the end of the last ice age?
- 5. Why is the average temperature of the oceans much colder than the average temperature of Earth's surface?
- 6. Why is Earth's lithosphere thin enough that it can break up into plates?
- 7. Why does the altitude of a mid-ocean ridge decrease as the square root of the age of the sea floor?

There are two ways in which heat is moved. If the thermally vibrating molecules pass their kinetic energy to adjacent molecules, we call the process "conduction" and more generally, when the process is time-dependent, we call it "diffusion". If heat is physically transported by larger-sale motion of a solid or fluid, we call the process "advection". If the motion resulting in advection is driven by the distribution of heat, we call it "convection".

Diffusion and advection are not limited to heat. For instance, viscous effects in fluids and the decay of electromagnetic fields in media with finite electrical conductivity lead to the same equations as the diffusion heat, but depend on different physical constants. Thus once we understand how diffusion and advection work for heat, we can use the same ideas to investigate other interesting phenomena such as:

- 8. Why do glaciers have an approximately parabolic variation of velocity across their width, while rivers have a more uniform velocity across their width?
- 9. How can charged particles in the solar wind enter Earth's magnetosphere?
- 10. Why do hill-shapes blur over time?
- 11. Why do tea leaves at the bottom of a cup that is stirred gather at the center of the cup when considerations of centrifugal force suggest that they should move to the walls of the cup?

Then, with relatively modest additions to the basic ideas of diffusion, we will be able to understand important aspects of the dynamics of Earth's ocean and atmosphere:

- 12. Why do ice floes drift in a different direction than the wind blowing over the surface of the ocean?
- 13. Why are large-scale ocean currents concentrated along the western edges of the oceans?
- 14. Why is there a jet stream in the atmosphere?

## Derivation of the diffusion equation

We begin by deriving the equation that governs the time-dependence of temperature in a thermally conducting medium. This derivation is not mathematically rigorous, but has the right physical ideas and gets the right answer. This leads to one of the most important partial differential equations of physics and gives us the opportunity to demonstrate how to get useful information from such an equation without actually solving it. How to derive the solution for the case of a sinusoidal temperature variation at the surface of a half space is given in an appendix to this chapter. You will study the behavior of this solution and the closely related case of a step change in the surface temperature in a lab.

Heat is a concept invented before scientists understood the molecular nature of materials. We now know that it is kinetic energy associated with the vibrational ``dance" that all molecules do at temperatures above absolute zero. This dance is random and when averaged over volumes that contain many molecules averages to zero. Thus it does not involve any net movement of material. On the other hand, if you place a ``hot" object with rapidly dancing molecules next to a ``cold" object with slowly dancing molecules, collisions at the boundary pass momentum from some of the fast molecules to some of the slow ones. The molecules in the cold material near the boundary will then have somewhat higher average energy than they did before and will appear to warm up. The molecules in the hot material near the boundary will have somewhat lower average energy than they did before and will appear to some the slow of the hot material near the boundary will have somewhat lower average energy than they did before and will appear to cool down. We say that heat has flowed from the hot material to the cold material.

We do not actually measure the kinetic energy of vibration. In any case, it would be too much bother to find out what each molecule is doing and then add them all up. Instead we measure the effect of the ensemble of vibrating molecules on something else such as the volume of mercury in a thermometer or the electrical resistance of carbon. Historically, these measurements have been reported as a quantity called temperature. There is a particular temperature scale (called the absolute or Kelvin scale) in which the temperature T is directly proportional to the heat (i.e kinetic energy of vibration) per unit volume

$$q = \rho c_p T$$

where  $\rho$  is the density and  $c_p$  is a constant called the "specific heat". We are only concerned with changes in heat and temperature so we write

$$\Delta q = \rho c_p \Delta T$$

Because T (°C) = T (°K) – 273.15,  $\Delta$ T (°C) =  $\Delta$ T (°K) and the last relation is valid for the more commonly used centigrade (Celsius) temperature with no changes. It can be converted to the Farenheit scale by re-defining the value of  $c_p$ . The MKS units of q are

the same as kinetic energy (joules) and the old units of heat (calories or Kilo-calories) are no longer recognized except in dietary information!

The first really quantitative study of heat flow was done by Fourier who conducted a series of careful experiments, which showed that the heat ``flux" (heat moving per unit time) through a linear rod of unit area

$$h = k \frac{\Delta T}{L}$$

where k is an experimental constant called the "thermal conductivity",  $\Delta T$  is the temperature difference between the ends of the rod and L is the length of the rod. This is known as Fourier's Law. The thermal conductivity could in principle be predicted using "statistical" mechanics. However, it is almost always simply measured.

More generally, if T varies only in the x direction, Fourier's Law becomes

$$h = -k\frac{dT}{dx}$$

and if T varies in multiple dimensions

$$h = -k \nabla T$$

The negative sign in the last two relations is there because heat flows "downhill" from hot to cold.

Now consider what happens when we increase the temperature by an amount  $\Delta T$  on one face of a rectangular box of material (assumed to be solid for the moment) whose volume is  $\Delta X \Delta Y \Delta Z$  and whose other faces are kept insulated.





Conservation of energy requires that the rate of change of heat inside the volume must equal the flux of heat into the volume. Thus if H is the total flux in, Q is the total heat inside and  $\Delta t$  is a unit interval of time, we must have

$$H = \frac{\Delta Q}{\Delta t}$$

Since we are only changing the temperature on one Y-Z face, the direction of heat flow is in the X direction. The length of the volume in the direction of heat flow is  $\Delta X$ , so the heat flux per unit area is approximately

$$h = k \frac{\Delta T}{\Delta X}$$

and the total heat flux through the face of the box is

$$H = h \Delta Y \Delta Z = k \frac{\Delta T}{\Delta X} \Delta Y \Delta Z$$

The total heat inside the volume is q times the volume. Thus the rate of change of Q is

$$\frac{\Delta Q}{\Delta t} = \frac{\Delta q}{\Delta t} \Delta X \Delta Y \Delta Z = \rho c_p \frac{\Delta T}{\Delta t} \Delta X \Delta Y \Delta Z$$

Equating this to the total heat flux we have

$$\rho c_p \frac{\Delta T}{\Delta t} \Delta X \Delta Y \Delta Z = k \frac{\Delta T}{\Delta X} \Delta Y \Delta Z$$

which can be re-arranged as

$$\frac{\Delta T}{\Delta t} = \frac{k}{\rho c_p} \frac{1}{\Delta X} \left( \frac{\Delta T}{\Delta X} \right)$$

Referring to our discussion of derivatives in *Mathematical Tools*, we see that as the size of the box and the time interval become very small, this equation becomes

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$$
  
where  $\kappa = \frac{k}{\rho c_p}$  is called the thermal "diffusivity".

This last equation is our first example of a "Partial Differential Equation" (PDE) because the temperature T(x, t) is a function of both space and time and the equation involves derivatives with respect to both arguments. This equation is called the one-dimensional (1D) diffusion equation. The equation prior to making the box very small is a "finite difference" approximation to the 1D diffusion equation.

### Approximate behavior of the solution of the 1-D diffusion equation

We now introduce a technique for figuring out the likely behavior of the solution to a partial or ordinary differential equation without solving it. The basic idea is very simple. Each derivative in the PDE is approximated by the expected change of the variable divided by an appropriate scale over which this change takes place. For instance, if temperature varies by  $\Delta T$  over a distance  $\delta$ , we would let

$$\frac{\partial T}{\partial x} \approx \frac{\Delta T}{\delta}$$

We do the same thing for a time variation and let

$$\frac{\partial T}{\partial t} \approx \frac{\Delta T}{\tau}$$

where  $\tau$  is a time scale over which significant temperature change occurs.

This technique should work because the solutions to most PDE's are well-approximated by an exponential. For a real exponential such as

$$F(x) = Ae^{\frac{x}{\delta}}$$

the derivative is

$$\frac{dF}{dx} = A\frac{d}{dx}\left(e^{\frac{x}{\delta}}\right) = \frac{A}{\delta}e^{\frac{x}{\delta}} = \frac{F}{\delta}$$

while for an imaginary exponential such as

$$G(t) = Ce^{i\omega t}$$

the derivative is

$$\frac{dG}{dt} = C \frac{d}{dt} \left( e^{i\omega t} \right) = i\omega C e^{i\omega t} = i\omega G = i \frac{G}{\tau}$$

where  $\tau$  is the "radian" period of the oscillation (period divided by  $2\pi$ ). The "i" on the right side shifts the phase of the oscillation by 90°, but does not change its magnitude. Thus, in both cases, the approximation we have made is likely to be pretty good in predicting the magnitude of the behavior of the PDE solution.

Suppose we want to know how the annual temperature variation diffuses into the Earth. Substituting the above approximations into the 1D diffusion equation we get

$$\frac{\Delta T}{\tau} \approx \kappa \frac{1}{\delta} \left( \frac{\Delta T}{\delta} \right)$$

Since we know the time scale in this situation, we solve this equation for what we do not know: the depth at which the temperature variation has substantially decayed.

$$\delta \approx \sqrt{\kappa \tau}$$

Note that  $\Delta T$  has cancelled out. Thus the shape of the depth variation of the solution does not depend on the amplitude of the temperature variation at the ground surface.

The molecular thermal diffusivity of rocks is about  $2 \times 10^{-7} \text{ m}^2/\text{s}$ . Since the annual temperature cycle is roughly sinusoidal and there are almost exactly  $\pi \times 10^7$  seconds in a year (a handy number to remember along with  $10^5$  seconds in a day), the radian period  $\pi \times 10^7/2\pi = 5 \times 10^6$  s. This gives  $\delta \approx 1$  meter. It is thus not surprising that the annual temperature variation has little effect on the cellar of a house. Furthermore, this scale is small compared to any horizontal length scale likely for the surface temperature variation, so we are justified in using the 1D diffusion equation.

## Enhanced diffusion due to mixing: Turbulence

The molecular thermal diffusivity for water is similar to that for rocks, so we could predict that the annual atmospheric temperature cycle should penetrate on order of a meter in an ocean. Here are vertical profiles of ocean summer and winter temperature in the NE Pacific at Ocean Weather Station (OWS) Papa at 50° N, 145 °W.



The annual cycle has affected a layer 100 m thick, not 1 m. This is because fluid motions are mixing the top 100 m of the ocean. Consider what happens if you start out with a straight boundary between two regions of uniform temperature and deform it by a rotary motion.



Before the motion starts, molecular diffusion at the interface occurs as we have already discussed and the time scale is the molecular diffusion time. The rotary motion does two things: (1) it advects warm fluid into the cold region and vice versa and (2) it creates thinner and thinner layers that alternate hot and cold. This second process greatly reduces the length scale over which molecular diffusion needs to act to smooth out the spatial temperature fluctuations. The combination of these two processes results in an apparent

diffusion of heat that is much faster than molecular diffusion without mixing. Mixing processes that arise spontaneously in fluids are called "turbulence" and it can be shown that the average of the temperature over time scales long compared to the fluid motion time, but short compared to the original molecular diffusion time scale satisfies the diffusion equation with an "eddy" diffusivity that is much larger than the molecular value. It is obvious from the last equation, that increasing  $\delta$  a factor of 100 for fixed  $\tau$  requires that  $\kappa_{eddy}$  be 10<sup>4</sup> times larger than its molecular value.

There, however, another important aspect of the vertical temperature that is not explained by turbulent mixing and, in fact cannot be explained by molecular diffusion. Here is a typical profile of temperature for the full depth of the ocean at low to mid-latitudes.



It can be divided into three layers: a nearly uniform top layer (called the mixed layer); a steep drop in temperature that extends to several hundred meters called the "main thermocline" (as opposed to the "seasonal thermocline" seen further north at OWS Papa) and a deep region extending to the full depth that has a nearly constant temperature several degrees above 0.

Re-arranging the above expression for  $\delta$  gives

$$\tau \approx \frac{\delta^2}{\kappa}$$

Using  $\kappa_{eddy} = 10^4 \kappa_{molecular} = 2 \times 10^{-3} \text{ m}^2/\text{s}$  and an ocean depth of 4 km, gives  $\tau = 8 \times 10^9 \text{ s} = 200$  years. So we could predict that the deep ocean should have the average temperature of Earth's surface averaged over the last several centuries. Clearly this is not so. The remarkably constant temperature below the the thermocline is substantially colder than the average temperature of Earth's surface.

One could suggest that turbulent diffusion is not effective below the upper 100 m. This is certainly part of the story, as we will see later. However, if we instead use the molecular diffusivity of water, we increase the time scale by a factor of  $10^4$ . Then  $\tau$  equals 2 million years. This is a much longer time scale, but is still very short compared to the age of Earth and thus presumably to the age of the oceans. There is certainly no reason to think that the average temperature of Earth's surface has been cold enough in recent geologic time (or in any geologic time) to account for the deep cold water in the oceans. So something else must be going on.

# 1D Diffusion in a moving material

There is an important subtlety that comes up when diffusion occurs in material is moving relative to the observer. The above diffusion equation must be understood as being from the point of view of a coordinate system (or observer) fixed to the box. That is, no material is allowed to cross the boundary of the box. The reason for this is that we require that heat get into the box only by conduction. If mass flows across the boundary it can carry material with a different temperature into the volume and our calculation is no longer correct. Many times, however, we want to know what happens to the time derivative of temperature at a position that is not attached to the material. For instance, we might want to know the rate of rise in temperature at a weather station as opposed to the temperature observed by a balloon floating by in the wind.

It turns out that a simple connection exists between the time derivatives in coordinate systems moving relative to one another. This figure illustrates the issue involved



The vertical boxes are slits though which observers see the value of a sine wave fixed to the page. One slit is fixed in space, the other moves to the right. The observer looking through the fixed slit sees a constant value. The observer looking through the moving slit sees an oscillating value. If the fixed observer knows the wavelength of the sinusoid and how fast the moving observer is traveling, he can easily calculate what the moving observer will see.

To make this concept quantitative, the total change in a function F(t, x) due to small changes  $\delta t$  and  $\delta x$  is

$$\delta F = \frac{\partial F}{\partial t} \,\delta t + \frac{\partial F}{\partial x} \,\delta x$$

Dividing this by  $\delta t$  we get

$$\frac{\delta F}{\delta t} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\frac{\delta x}{\delta t} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x}$$

The term on the left is the total rate of change of F is some coordinate system. The first term on the right is the rate of change of F holding position constant in this coordinate system. The second term on the right involves the velocity

$$u = \frac{\delta x}{\delta t}$$

at which position is changing with respect to the same coordinate system. Thus it is the rate of change of F due to moving by the spatial gradient of F at velocity u.

If we identify the original coordinate system with one fixed to the material and use the notation introduced in Mathematical Tools, we can relate the rate of change of temperature as seen by an observer fixed to material moving at velocity u to the rate of change of temperature seen by an observer fixed in space by

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x}$$

The second term on the right is called the "advection". The term on the left is called the "substantial" derivative because it is relative to the "substance". In order to write the 1D diffusion equation for moving material viewed in a fixed coordinate system (or fixed material viewed from a moving system), all we need to do is replace the partial derivative with respect to time with the substantial giving

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = \kappa \frac{\partial^2 T}{\partial x^2}$$

#### Maintaining the oceanic thermocline

We have shown thermal diffusion in the shallow ocean is much too fast to account for a deep ocean that is much colder than the average surface temperature. This means that we need a process that can counter the downward diffusion. Upward advection of cold water is an obvious candidate. If the thermocline is a stable feature of the ocean, we can set the time derivative term in the 1D diffusion equation with advection to zero, we then have

$$w\frac{\partial T}{\partial z} = \kappa \frac{\partial^2 T}{\partial z^2}$$

As we will consistently do in this course, the vertical coordinate is z and the vertical velocity is w. If we approximate the vertical derivatives of T as before, we can easily show that

$$w = \frac{\kappa \left(\frac{\Delta T}{\delta^2}\right)}{\left(\frac{\Delta T}{\delta}\right)} = \frac{\kappa}{\delta}$$

An appropriate value for  $\delta$  is the thickness of the region of high temperature gradient. This is about 500 m in the above figure. The vertical velocity required to keep the thermocline stable in the face of the downward turbulent diffusion is then

$$w = \frac{\kappa_{eddy}}{500} = \frac{2 \times 10^{-3} \ m^2/s}{500 \ m} = 4 \times 10^{-6} \ m/s \approx 100 \ m/y$$

This velocity is actually an upper bound, because the eddy diffusivity almost certainly decreases at depth as we get further from the active processes like wind waves at the top of the ocean. However, this value is already so low that it cannot be directly measured against the background of other oceanic motions.

### Appendix

## The 3D diffusion equation in material at rest

If there are thermal gradients in all directions, a similar, but more involved argument leads to the three-dimensional (3D) diffusion equation

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T$$

where

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla \cdot \nabla T$$

is the "Laplacian" of the temperature. If the diffusivity  $\kappa$  is not a constant in space, but is isotropic (does not vary with direction), it has to be taken inside the spatial derivative and the right side of the 3D diffusion equation becomes  $\nabla \cdot \kappa \nabla T$ . Things become considerably more complicated if the diffusivity is anisotropic (depends on the direction of the temperature gradient). This can happen, for instance, in a sediment which has layers of good thermal conductor separated by layers of bad thermal conductor. Such situations are way beyond the scope of this course!

#### 3D Diffusion in a moving material

The argument outlined is easily generalized to a vector velocity  $\mathbf{u}$  and spatial gradients of T in any direction. The result is

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa \nabla^2 T$$

Non-dimensionalization of the 1D diffusion equation

This section presents the solution for the 1D diffusion equation in material at rest when the surface temperature has a sine variation in time. We begin with a non-essential but useful step called non-dimensionalization. Let  $\Delta T$ ,  $\tau$  and  $\delta$  be a constant temperature variation scale, a constant time scale and a constant length scale. Then define nondimensional temperature, time and length variables

$$T' = \frac{T}{\Delta T}$$
  $t' = \frac{t}{\tau}$   $z' = \frac{z}{\delta}$ 

which can also be re-arranged as

$$T = (\Delta T)T'$$
  $t = \tau t'$   $z = \delta z'$ 

Using the rules for differentiation in *Mathematical Tools* 

$$\frac{\partial T}{\partial t} = \frac{\partial t'}{\partial t} \frac{\partial T}{\partial t'} = \frac{1}{\tau} \frac{\partial T}{\partial t'} = \frac{1}{\tau} \frac{\partial}{\partial t'} \Big[ (\Delta T) T' \Big] = \frac{\Delta T}{\tau} \frac{\partial T'}{\partial t'}$$

Exactly the same steps give

$$\frac{\partial T}{\partial z} = \frac{\Delta T}{\delta} \frac{\partial T'}{\partial z'}$$

and thus

$$\frac{\partial^2 T}{\partial z^2} = \frac{\partial}{\partial z} \left( \frac{\partial T}{\partial z} \right) = \frac{\partial}{\partial z} \left( \frac{\Delta T}{\delta} \frac{\partial T'}{\partial z'} \right) = \frac{\partial z'}{\partial z} \frac{\partial}{\partial z'} \left( \frac{\Delta T}{\delta} \frac{\partial T'}{\partial z'} \right) = \frac{1}{\delta} \frac{\partial}{\partial z'} \left( \frac{\Delta T}{\delta} \frac{\partial T'}{\partial z'} \right) = \frac{\Delta T}{\delta^2} \frac{\partial^2 T'}{\partial z'^2}$$

Using these results, the dimensional form of the 1D diffusion equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2}$$

becomes

$$\frac{\Delta T}{\tau} \frac{\partial T'}{\partial t'} = \frac{\kappa(\Delta T)}{\delta^2} \frac{\partial^2 T'}{\partial z'^2}$$

which can be simplified to

$$\frac{\partial T'}{\partial t'} = \left[\frac{\kappa\tau}{\delta^2}\right] \frac{\partial^2 T'}{\partial z'^2}$$

This may seem like a lot of work for little gain. The only useful thing that has obviously happened is that the temperature variation scale has canceled out. However, the time and length scales are not yet chosen. If we chose them cleverly we can make

$$\left[\frac{\kappa\tau}{\delta^2}\right] = 1$$

and then the non-dimensional form of the PDE becomes

$$\frac{\partial T'}{\partial t'} = \frac{\partial^2 T'}{\partial z'^2}$$

This is clearly a step forward, because it no longer depends on the diffusivity or the time and length scales. It can be solved without knowing anything about the actual material properties or dimensions of the problem.

Making the non-dimensional combination in square brackets equal one, however, implies that the time and length scales of the dimensional solution are not independent of each other. Given the diffusivity and the time scale, the length scale must be

$$\delta = \sqrt{\kappa \tau}$$

Alternatively, given the diffusivity and the length scale, the time scale must be

$$\tau = \frac{\delta^2}{\kappa}$$

This relationship is, of course, what we deduced from the approximate argument earlier in this chapter.

### Solution of the non-dimensional 1D diffusion equation

In most of the rest of this *Appendix*, we will drop the "primes" from the non-dimensional variables because it will be obvious from the context whether we are dealing with dimensional or non-dimensional quantities and the primes clutter up the notation. We therefore want to solve the PDE

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial z^2}$$

The specific problem we want to solve is the decay of the temperature variation

$$T(t) = \Delta T \sin(\omega t) = \Delta T \sin(\frac{t}{\tau})$$

applied to the top surface (z = 0) of a half space with constant diffusivity  $\kappa$ . The radian frequency  $\omega$  is the oscillation period divided by  $2\pi$  and the "radian period"

$$\tau = \frac{1}{\omega}$$

Because the PDE is  $2^{nd}$  degree, we require a second boundary condition. On physical grounds, we expect the temperature to decay with depth (z < 0) in the half space. Thus an appropriate second boundary condition is simply

$$T(t,z \rightarrow -\infty) = 0$$

Dividing the surface boundary condition by  $\Delta T$  and introducing the same nondimensional variables we used for the PDE, we can easily write it in its non-dimensional form:

$$T(t,0) = \sin(t)$$

(the primes have been dropped). The boundary condition at infinite is the same written in dimensional or non-dimensional variables.

The algebra of this problem is much easier if we solve the problem with the surface boundary condition

$$T(t,0) = e^{it}$$

We can then recover what we really want by taking the "imaginary part" at the end.

Whenever you have a PDE in which the derivatives in space and time are confined to separate terms, a useful trick is to let

$$T(t,z) = F(t)G(z)$$

Then

$$\frac{\partial T}{\partial t} = G \frac{dF}{dt}$$

and

$$\frac{\partial^2 T}{\partial z^2} = F \frac{d^2 G}{dz^2}$$

Ordinary derivatives appear on the right side of these equations, because F and G are functions of a single variable. Substituting these into the diffusion equation and dividing by FG gives

$$\frac{1}{F(t)}\frac{dF(t)}{dt} = \frac{1}{G(z)}\frac{d^2G(z)}{dz^2} = \alpha$$

The left side of this equation depends only on t and the right side depends only on z. The two sides can only be equal to each other if they are both equal to a constant, which we have called  $\alpha$ . Therefore

$$\frac{dF}{dt} = \alpha F \qquad \qquad \frac{d^2G}{dz^2} = \alpha G$$

The original PDE with  $1^{st}$  and  $2^{nd}$  partial derivatives has been replaced by two ordinary differential equations, one of  $1^{st}$  and the other of  $2^{nd}$  degree. This technique is called "separation of variables" and  $\alpha$  is called the "separation constant". Since the surface boundary conditions is

$$T(t,0) = F(t)G(0) = e^{it}$$

we can choose

G(0) = 1

and then the surface condition on F is

 $F(t) = e^{it}$ 

The boundary condition for the separated problem at infinity is

$$G(-\infty) = 0$$

We know from *Mathematical Tools* that the derivative of an exponential is an exponential and so the solution to the ODE for F is going to be an exponential. In fact, we saw this ODE in the atmospheric pressure example in *Mathematical Tools* and so we know that its solution is

$$F(t) = A e^{\alpha t}$$

It is clear from the boundary conditions that A = 1 and  $\alpha = i$ .

The ODE for G is then

$$\frac{d^2G}{dz^2} = iG$$

We know from *Mathematical Tools* that the 2<sup>nd</sup> derivative of an exponential is an exponential, so the solution to this ODE is also going to be an exponential. Substituting

$$G(z) = e^{\beta z}$$

in the ODE for G gives

$$\beta^2 G = i G$$

and therefore (using a result derived in *Mathematical Tools*)

$$\beta = \pm \sqrt{i} = \pm \frac{1+i}{\sqrt{2}}$$

There are consequently two solutions

$$G(z) = Be^{-\left(\frac{1+i}{\sqrt{2}}\right)^{z}} = Be^{-\frac{z}{\sqrt{2}}}e^{-\frac{z}{\sqrt{2}}}$$

and

$$G(z) = C e^{+\left(\frac{1+i}{\sqrt{2}}\right)^{z}} = C e^{+\frac{z}{\sqrt{2}}} e^{+\frac{z}{\sqrt{2}}}$$

Since z is less than zero, the first of these solutions blows up as depth increases. This is not permitted by the boundary condition, so we must make B = 0. The surface condition then obviously requires C = 1 and we have

$$T(t,z) = F(t)G(z) = e^{it} e^{+\frac{z}{\sqrt{2}}} e^{+\frac{z}{\sqrt{2}}} = e^{\frac{z}{\sqrt{2}}} e^{i\left(t+\frac{z}{\sqrt{2}}\right)} = e^{\frac{z}{\sqrt{2}}} \left(\cos\theta + i\sin\theta\right)$$

where

$$\theta = t + \frac{z}{\sqrt{2}}$$

is called the "phase" of the solution. Finally, because we only want the sine variation, we take the imaginary part of this result giving

$$T(t,z) = e^{\frac{z}{\sqrt{2}}} \sin\theta$$

The magnitude of this solution decays exponentially as z becomes more negative we expect. It equals  $e^{-1}$  when  $z = -\sqrt{2}$ . Its "skin depth" in dimensional variables is therefore,

$$z_{skin} = \delta \sqrt{2} = \sqrt{2} \sqrt{\kappa t} = \sqrt{2\kappa\tau}$$

If we had been more clever in our non-dimensionalization and used the length scale

$$\delta = \sqrt{2\kappa\tau}$$

we could avoided all the factors of  $\sqrt{2}$  in the non-dimensional solution.

Our approximate analysis earlier missed the  $\sqrt{2}$  factor, but since we never expected it to be more than an order of magnitude result, this is not an important flaw. A more serious

flaw in our approximate analysis is that it missed the fact that the phase of the solution changes with depth. At the surface, the first zero-crossing of the solution is at t = 0. At the skin-depth, however, the phase of the solution at the same time is

$$\theta = t + \frac{z}{\sqrt{2}} = 0 + \frac{-\sqrt{2}}{\sqrt{2}} = 1$$

radian. To find the time of the first zero-crossing, we need to solve

$$\theta = t + \frac{z}{\sqrt{2}} = t + \frac{-\sqrt{2}}{\sqrt{2}} = 0$$

We obtain a non-dimensional time of t = 1 which is a dimensional time of  $\tau$ . Thus the temperature sine wave at the skin depth is delayed by the radian period and reduced in amplitude by 1/e. Because the sine wave is delayed in time, we say that it has a "phase lag". This phase lag increases linearly with depth. It is  $2\tau$  at two skin depths and  $3\tau$  at three skin depths. The temperature signal is flipped upside down when the phase lag is  $\pi$  (180 degrees), which occurs when  $z = \pi\sqrt{2} = 4.44$  times the skin depth.

Here are plots of non-dimensional magnitude as a function of non-dimensional depth in linear and semi-log format



and here is a plot of the phase lag (which is non-dimensional) as a function of nondimensional depth

