ESS314

Basics of Geophysical Fluid Dynamics by John Booker and Gerard Roe

Conservation Laws

The big differences between fluids and other forms of matter are that they are continuous and they deform internally and continuously if shearing forces are present. Physical laws for fluids are the same as for all matter, but are most conveniently written in terms of quantities per unit volume. The conservation of momentum is thus

$$\mathbf{f} = \rho\,\mathbf{a} = \rho\frac{D\mathbf{u}}{Dt} = \frac{D^2\mathbf{r}}{Dt^2}$$

where $\mathbf{r}$ is the instantaneous position of a "fluid particle" and ρ is the fluid density. As we have already discussed, a tricky point is that the volume of fluid used in this law is attached to the molecules of the fluid. No mass is allowed to cross its surface. The shape of the volume can deform as the fluid moves, but the mass inside the volume is constant. To remind us that we are discussing a volume that moves with the fluid, we have replaced the *d* in the time derivative by *D*.

Accelerations in moving reference frames

In the section on *Diffusive Phenomena*, we showed that the relation between time derivatives in frames moving at velocity u in the x-direction relative to one another is

$$\frac{D}{Dt} = u\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$$

The 1st term on the right is the apparent time derivative due to sweeping spatial variations by the observer. We also noted in the *Appendix* to the same section that this can be generalized to motion in three-dimensions:

$$\frac{D}{Dt} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + \frac{\partial}{\partial t} = (\mathbf{u}\cdot\nabla) + \frac{\partial}{\partial t}$$

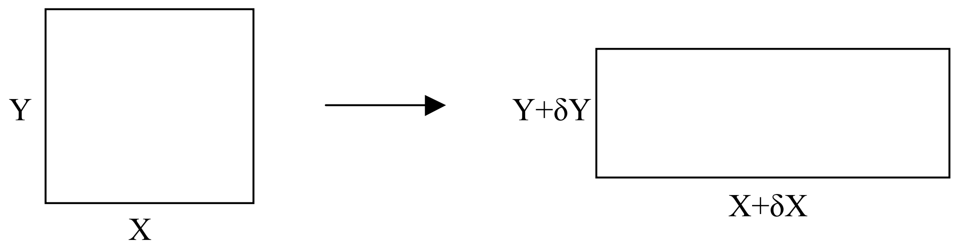
This "operator" can be applied to a vector as well as a scalar. For velocity, $\mathbf{u}$, we get

$$\frac{D\mathbf{u}}{Dt} = (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \frac{\partial\mathbf{u}}{\partial t}$$

The term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ is non-linear and is responsible for turbulence and other complications in fluid dynamics. Fortunately, it is identically zero when $\mathbf{u}$ does not change in the direction of $\mathbf{u}$ (i.e. $\mathbf{u}$ is perpendicular to $\nabla\mathbf{u}$). Furthermore, in important applications, such as waves, $(\mathbf{u}\cdot\nabla)\mathbf{u}$ can be small enough to ignore. So we can investigate many phenomena without dealing with this difficult term.

The effect of mass conservation on the velocity field

If a fluid is incompressible (i.e. ρ is constant for a volume moving with the fluid, but not necessarily the same in different volumes), conservation of mass implies conservation of volume. The effect of this on the velocity is easier to understand for 2D velocity $\mathbf{u} = u\hat{\mathbf{x}} + v\hat{\mathbf{y}}$. Then the z-dimension of the fluid is constant and conservation of volume implies conservation of area in (x, y) planes. Let a rectangle with sides X and Y be deformed to a rectangle with sides X+δX and Y+δY.



Conservation of area implies

$$X\,Y = (X + \delta X)(Y + \delta Y) = X\,Y + Y\delta X + X\delta Y + \delta X\delta Y$$

If the rectangle is small, δXδY is small compared to the other terms and can be ignored. Then this equation becomes

$$X\,Y = X\,Y + Y\delta X + X\delta Y$$

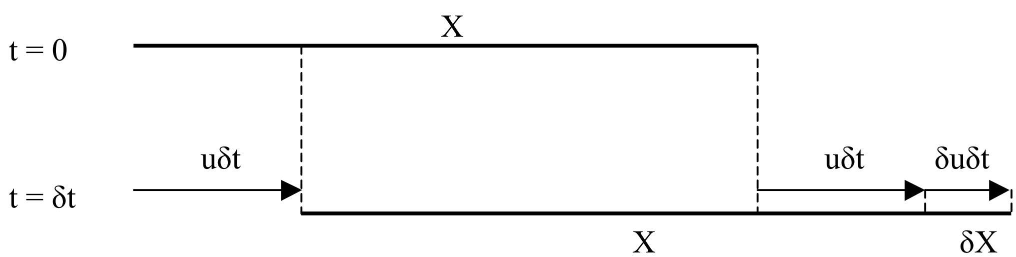
which implies that

$$Y\delta X = -X\delta Y$$

Dividing by the original area XY, we get

$$\frac{\delta X}{X} = -\frac{\delta Y}{Y}$$

Thus conservation of area implies that relative stretching of the rectangle in the x-direction must be balanced by relative shrinking in the y-direction. Now consider what happens to the X dimension of the rectangle if the x-velocity at the left side of the rectangle u increases to u+δu at the right side. After a time δt, the right side has moved more than the left side. From this figure:



it is clear that the length of the X-dimension has been stretched by the amount: $\delta X = \delta u\,\delta t$. If u is increasing at a constant rate $\frac{\partial u}{\partial x}$ in the x–direction, $\delta u = X\frac{\partial u}{\partial x}$

and then

$$\delta X = \delta u \delta t = X\frac{\partial u}{\partial x}\delta t$$

Therefore

$$\frac{\delta X}{X} = \frac{\partial u}{\partial x}\delta t$$

This result is valid for more complicated spatial derivatives of u if X is small. Exactly the same argument for a variation of the y-velocity in the y-direction leads to

$$\frac{\delta Y}{Y} = \frac{\partial v}{\partial y}\delta t$$

Adding these last two equations gives

$$\frac{\delta X}{X} + \frac{\delta Y}{Y} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\delta t$$

Conservation of area implies that the left side of this equation is zero and hence

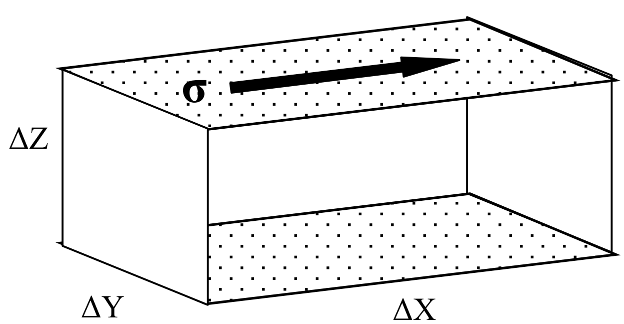
$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

which is the 2D form of $\nabla \cdot \mathbf{u} = 0$. This argument can easily be generalized to 3D and the velocity field in an incompressible fluid must obey

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \nabla \cdot \mathbf{u} = 0$$

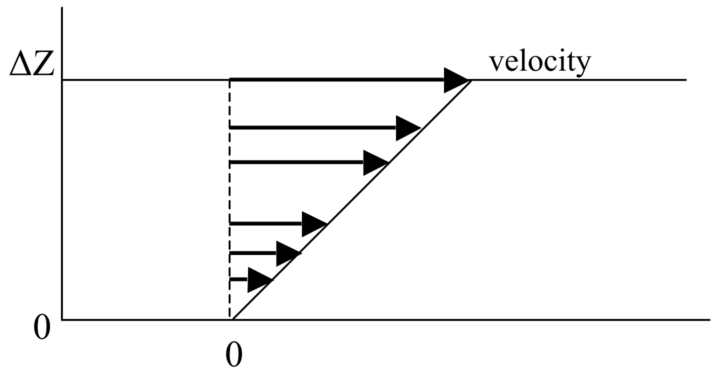
Viscous forces

We now want to enumerate the forces that act on the left side of Newton's 2nd Law. Consider first frictional forces, usually termed "viscous" in the context of fluids. Shear stress is defined as the force per unit area exerted tangentially on a surface. This figure shows a horizontal slab of material with a shear stress $\boldsymbol{\sigma}$ applied on the top in the $\hat{\mathbf{x}}$ direction.



If this slab is elastic and we fix the position of its bottom, experiments show that it will shear an amount that depends on a material property called the "shear modulus" and then stop. The material

resumes its original shape when **σ** is removed. A fluid, however, will continue to shear as long as **σ** is applied. A fluid cannot be at rest if it is subject to shear stress.
In fact, after some starting transients, the velocity settles down to time-independent velocity with a uniform gradient in the z-direction:



The magnitude of the shear stress on the top boundary and the vertical derivative of the velocity are observed to be related by

$$|\boldsymbol{\sigma}| = \sigma = \mu \frac{\partial u}{\partial z}$$

The parameter μ in this "shear-stress-strain-rate" relation is called the "dynamic viscosity" and has units of Pascal-s. It is the physical manifestation of friction between parts of the fluid that are moving at different velocities.

For "Newtonian" fluids, μ is independent of σ and $\partial u/\partial z$. Liquid water, air and honey with no sugar crystals are Newtonian to a very good approximation. Ice, ketchup and rocks are can be non-Newtonian. They all get "softer" (i.e. the viscosity drops) when the velocity gradient (i.e strain rate) increases. An approximation that is often used is

$$\frac{\partial u}{\partial z} = A\sigma^n$$

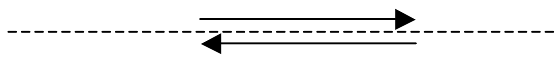
Fluids that obey this relation are called "power-law" fluids. Newtonian fluids are the case where n = 1 and A = 1/μ. At the conditions prevailing in glaciers, n is expected to be close to 3.

Both the shear stress and the strain-rate in the above flow are constant. So one can define an effective viscosity

$$\mu_{effective} = \frac{\sigma}{\left(\frac{\partial u}{\partial z}\right)}$$

which will be constant for the particular flow geometry in the above figure.

Each layer of the fluid applies the same tangential force to the layer just below. The fluid at the base of the slab applies this shear stress to the bottom boundary. To keep the fluid at the lower boundary from moving, it is necessary that the boundary apply an equal, but oppositely directed force to the base of the fluid. This illustrates an important point about shear stresses: Their direction depends on which side of a plane in the fluid you are looking at. The two force vectors acting tangentially on the plane indicated by the dashed line in this figure



result in the same sense of shear. If they were in the same direction, there might be a constant translation to the right, but no shear. Thus a shear stress applied in the $-\hat{\mathbf{x}}$ direction to the bottom of a plane is completely equivalent to a shear stress applied in the $+\hat{\mathbf{x}}$ direction to its top.

The uniform shear flow above has no time-dependence and is therefore not accelerated. In order for the fluid motion to change with time (i.e. accelerate), it is necessary that the shear stress at the top of the slab be unequal to the shear stress at the bottom. Let the shear stress be zero at the base and $\Delta\sigma$ at the top. The total force applied to the top and bottom of the slab is $\Delta\sigma\Delta X\Delta Y$ because there is no contribution from the bottom. The force per unit volume in the x direction is

$$f_x = \frac{\Delta\sigma\,\Delta X\Delta Y}{\Delta X\Delta Y\Delta Z} = \frac{\Delta\sigma}{\Delta Z}$$

As the size of the rectangular region slab becomes very thin, this becomes

$$f_x = \frac{\partial\sigma}{\partial z}$$

The 1D conservation of momentum equation becomes

$$\rho\frac{Du}{Dt} = \frac{\partial\sigma}{\partial z}$$

If the fluid is Newtonian, we can use the shear-stress-strain-rate relation to eliminate σ giving

$$\rho\frac{Du}{Dt} = \frac{\partial}{\partial z}\mu\frac{\partial u}{\partial z}$$

If μ is constant and we divide by the density we get

$$\frac{Du}{Dt} = \frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2} = \nu\frac{\partial^2 u}{\partial z^2}$$

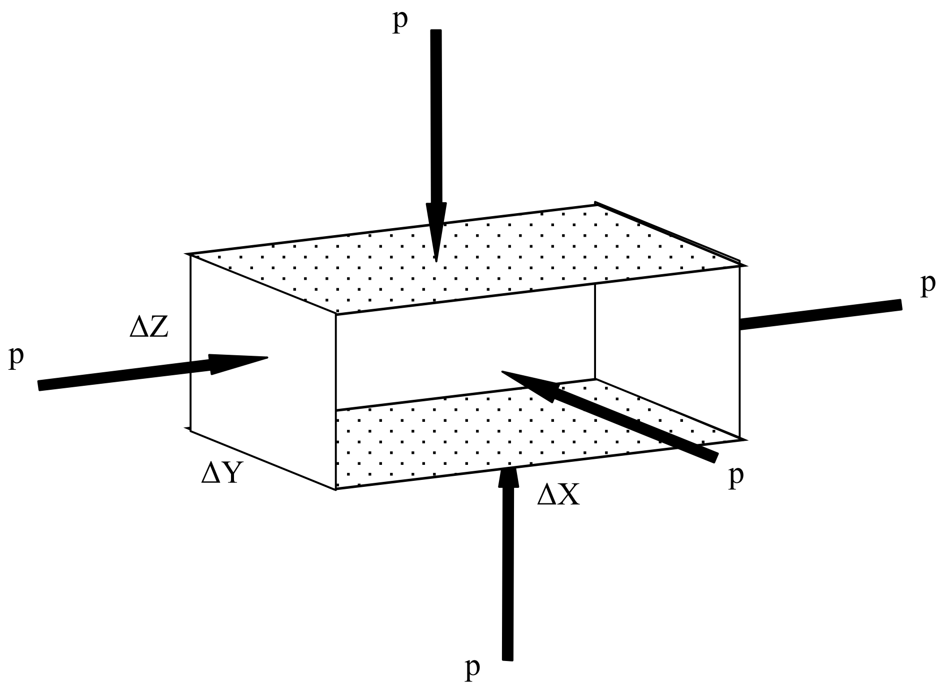
You should recognize this as the 1D diffusion equation. The “kinematic viscosity”

$$\nu = \frac{\mu}{\rho}$$

is the diffusivity of momentum. It is 10^{-6} m^2/s for water. The dynamic viscosity of air is much less than for water, but its density is proportionately less also and so the kinematic viscosity of air and water are quite similar.

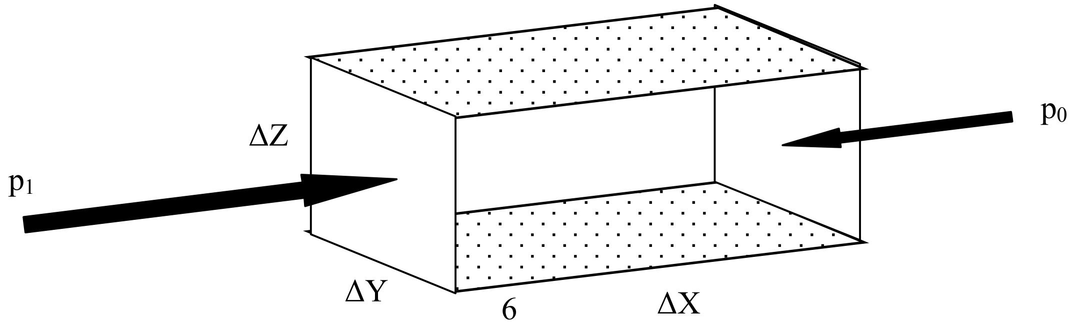
The pressure gradient force

Friction acts tangentially to surfaces in a fluid. Force per unit area perpendicular to a fluid surface it is called "pressure". Consider the same block of material as before. If we apply pressure p equally on all faces,



the volume of the fluid will decrease if it is compressible. It will expand to its old size if the pressure is removed. This is true whether the material is elastic or a fluid. If the material is in-compressible, the volume will not change in the presence of pressure. In either case, however, there is no net force to move the volume. Thus, unlike shear stress, pressure can exist in a fluid at rest.

Now suppose that the pressure on the two Z-Y faces are not equal.



To make the picture simpler, the average pressure has been subtracted. This makes no difference because, as we have just discussed, the average pressure acting equally in all directions does not lead to motion. The total force on the left-hand Z-Y face of the block is $p_1\Delta Y\Delta Z$ and the total force on the right-hand side of the block is $p_0\Delta Y\Delta Z$. The net total force per unit volume in the x-direction is

$$f_x = \frac{p_1\Delta Y\Delta Z - p_0\Delta Y\Delta Z}{\Delta X\Delta Y\Delta Z} = \frac{(p_1 - p_0)\Delta Y\Delta Z}{\Delta X\Delta Y\Delta Z} = \frac{\Delta p}{\Delta X}$$

and if the x-dimension of the block becomes small

$$f_x = -\frac{\partial p}{\partial x}$$

The negative sign is because the block has a net push to the right (increasing x) when the pressure is decreasing to the right. This is called the "pressure-gradient force". Adding it into the 1D Newtonian conservation of momentum equation we get

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial z^2}$$

In 3D this becomes

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \mu\nabla^2\mathbf{u}$$

Body Forces

The friction and pressure-gradient forces act fundamentally on surfaces and were converted to forces acting throughout the fluid. There are forces, however, that act fundamentally throughout the interior of the fluid. The most obvious example is the gravitational force per unit volume $\rho\mathbf{g}$. Including this in the 3D conservation of momentum equation in the form that we will most commonly want to use it

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p - g\hat{\mathbf{z}} + \nu\nabla^2\mathbf{u}$$

We have divided by density and defined the direction of gravity to be vertical with z positive upward. Other body forces are present if the fluid is electrically charged and is subject to an electric field or if it is electrically conducting and moves in a magnetic field. We have discussed these forces in the context of plasmas, but will not go further into them in this course.

Density and pressure variation in an ocean and atmosphere at rest

If the velocity is zero, the momentum equation reduces to

$$\nabla p = -\rho g\hat{\mathbf{z}}$$

Its horizontal components

$$\frac{\partial p}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial p}{\partial y} = 0$$

imply that p depends only on the vertical coordinate z. If ρ is constant, its vertical component

$$\frac{dp}{dz} = -\rho g$$

can be integrated to give

$$p = p_0 - \rho g z$$

where p_0 is the pressure at z = 0. This relation is adequate in the ocean where this "hydrostatic" pressure increases linearly with depth by about 1 atmosphere for every 10 meters and in the solid earth, where it is called "lithostatic" pressure and increases about 1 atmosphere every 3 meters.

Earth's atmosphere, however, is a compressible mixture of gasses and density depends on significantly on pressure. The ideal gas law can be written in the form

$$p = \rho RT$$

where R = 287 J/Kg/ºK for Earth's dry atmosphere and T is the absolute temperature. Although T varies by 10's of degrees laterally and vertically, these variations are small compared to the average temperature of the lower atmosphere, which is about 290 ºK. It is therefore reasonable to approximate T as constant. The z derivative of the ideal gas law is then

$$\frac{dp}{dz} = \frac{d\rho}{dz} RT$$

Using this to eliminate pressure derivative from the vertical component of the momentum equation, we obtain

$$\frac{d\rho}{dz} = -\left(\frac{g}{RT}\right)\rho = -\frac{\rho}{H}$$

where

$$H = \frac{RT}{g} \approx 8.5\, km$$

is the atmospheric "scale height" This ODE was solved in the *Mathematical Tools* section with the result that

$$\rho = \rho_0 e^{-\frac{z}{H}}$$

Alternatively, we can use the ideal gas law to eliminate ρ from the vertical component of the momentum equation to obtain

$$\frac{dp}{dz} = -\left(\frac{g}{RT}\right)p = -\frac{p}{H}$$

which is the same ODE as for ρ. Thus the vertical variation of pressure is identical to the vertical variation of density.

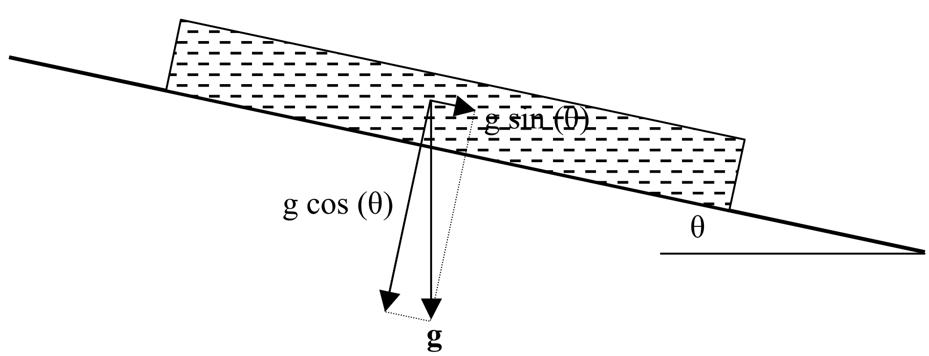
These results can be used to estimate the error we made above in approximating the density of the ocean as constant. We know the ocean is, in fact compressible because sound propagates at a speed of about c_{sound} = 1,500 m/s. One can argue on dimensional grounds alone that the scale height in the ocean should be

$$H = \frac{c_{sound}^2}{g} = \frac{(m/s)^2}{m/s^2} = m = \frac{(1500)^2}{9.8} = 230\ km$$

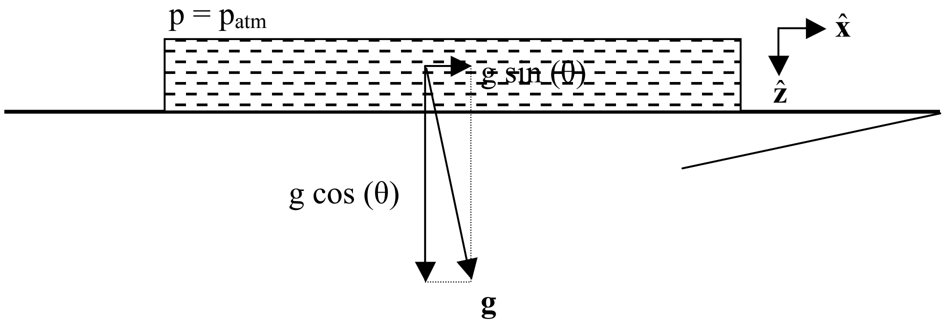
(This result follows rigorously from a derivation of sound speed from fluid compressibility.) Thus the compression increases density by a factor of $e^{(5/230)}$ = 1.02 at the bottom of the ocean over what it would be if the ocean were incompressible.

Flow of an ice sheet or glacier down a slope

Consider an ice sheet or glacier of constant thickness h on a slope that is inclined at an angle θ to the horizontal



It is clear that gravity has components along and perpendicular to the ice sheet. The gravity component along the ice sheet drives flow downhill. This downhill force must be balanced by shear stress at the base of the ice. This problem is easier to solve if it is rotated counter-clockwise so that $\hat{\mathbf{z}}$ is perpendicular to the top and bottom of the ice instead of in the direction of gravity. The algebra is also much simpler if we choose z = 0 at the upper surface of the ice and z = +h at the contact with the ground. Thus z increases downwards in contrast to almost all other examples we do in this course. This changes the sign of vertical derivatives in the momentum equation.



Note that with the choice of coordinates, the components of gravity are in the positive x and positive z-directions.

We make four assumptions: (1) the flow is horizontal in this rotated coordinate system; (2) variations in the y-direction can be neglected; (3) the flow is steady and (4) the ice is a power-law fluid. The first assumption is reasonable if the horizontal length of the ice sheet is long compared to its thickness and we stay away from the ends. The second is reasonable if the width of the flow is large compared to its thickness. Since ice sheets are typically 100's of km in both length and width and only 3 km or less thick, these assumptions are clearly reasonable. Glaciers are typically 10's of kilometers long and kilometers wide, but are only 10's of meters thick. Thus they also qualify for a 1D flow approximation.

The third assumption is reasonable if the time scale for velocity variations is long compared to the time it takes momentum to diffuse through the thickness of the ice. To estimate the viscous diffusion time for non-Newtonian ice, we need to use an effective viscosity. For glaciers and ice sheets it ranges range from 10^{10} to 10^{14} m^2/s. The maximum diffusive time scale is obtained using the minimum viscosity and the thickest ice. For 3 km thick ice

$$\tau_{\max} \approx \frac{\delta^2}{\nu} = \frac{(3000)^2}{10^{10}} \approx 10^{-3}\ s$$

Since this is many orders of magnitude less the years to centuries over which ice sheet and glacier motion varies, the second assumption is also entirely reasonable.

With these assumptions, the flow is 1D and the time-dependence of the momentum can be ignored. The vertical component of the momentum equation is

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g\cos(\theta)$$

or

$$\frac{\partial p}{\partial z} = \rho g \cos(\theta)$$

The right side of this equation is a constant. So its indefinite integral of with respect to z is

$$p(x,z) = z\rho g \cos(\theta) + B(x)$$

as you can easily verify by substitution. The pressure at the top of the ice (z = 0) is atmospheric (p_{atm}). The vertical variation of pressure in the atmosphere is very small compared to the vertical variation of pressure in ice, so to a very good approximation p_{atm} is independent of x. Therefore

$$B(x) = p_{atm}$$

is independent of horizontal position and

$$p(x,z) = p(z) = p_{atm} + z\rho g \cos(\theta)$$

is also. This pressure is, of course, equal to the hydrostatic pressure that would be exist if the ice were stationary.

The x-component of the 1D momentum equation is

$$0 = -\frac{\partial p}{\partial x} + \rho g \sin(\theta) - \frac{\partial \sigma}{\partial z}$$

where we have used the form that does not assume a Newtonian fluid. Note that the sign of the shear stress derivative has reversed because we have reversed the direction of increasing z.

We have just seen that p does not depend on x, so the pressure gradient term is zero and therefore

$$\frac{\partial \sigma}{\partial z} = \rho g \sin(\theta)$$

Again, the right side of this ODE is constant. Its indefinite integral is

$$\sigma(z) = z\rho g \sin(\theta) + C$$

Since the atmosphere exerts no shear stress on the top of the ice (not quite true, but wind stress is very small compared to the shear stresses in the much more viscous ice), C must be zero and

$$\sigma(z) = z\rho g \sin(\theta)$$

This the shear stress that the fluid above depth z exerts on the fluid below depth z. This shear stress is in the positive x-direction as it should be for a driving force in that direction. The ice below depth z resists the motion with a shear stress of the same magnitude but opposite, negative x-direction.

The next step is to substitute this shear stress into the shear-stress-strain-rate relation for a power law fluid:

$$-\frac{\partial u}{\partial z} = A\sigma^n = A[\rho g \sin(\theta)]^n z^n$$

Again, the negative sign on the left the consequence of reversing the direction of increasing z. It is useful to non-dimensionalize u and z by letting u = Uu' and z = hz'. Substituting these into the last equation and dropping the primes, we get

$$\frac{\partial u}{\partial z} = \left[-\frac{Ah^{n+1}}{U}[\rho g \sin(\theta)]^n\right] z^n = K z^n$$

where K is a constant that absorbs all the terms in front of z^n. The indefinite integral of this equation is

$$u = \frac{K}{n+1} z^{n+1} + D$$

The constant D is determined by the condition that u = 0 at z = 1 (i.e. the bottom of the ice at dimensional z = h). Thus

$$0 = \frac{K}{n+1}(1)^{n+1} + D$$

which can be solved for

$$D = \frac{-K}{n+1}$$

Then

$$u = \frac{-K}{n+1}[1 - z^{n+1}]$$

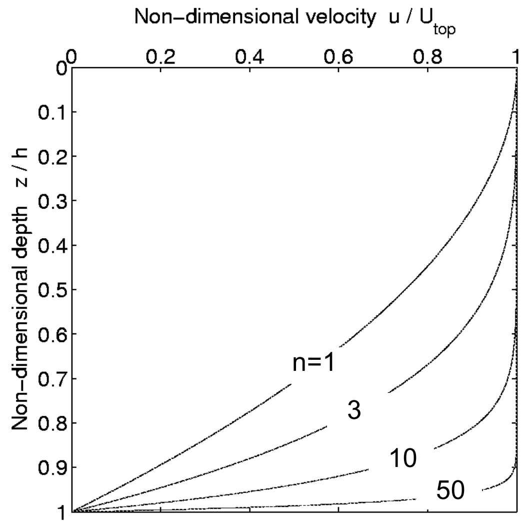
We have not yet specified the velocity scale U. This can be conveniently done by choosing

$$\frac{-K}{n+1} = 1$$

because then

$$u = [1 - z^{n+1}]$$

which is 1 at the top and 0 at the bottom of the ice. The following figure shows this non-dimensional function for n = 1 (Newtonian), 3 (glacier ice), 10 and 50.



As n increases, the shearing is confined more and more to the lower boundary of the ice. In the limit that n goes to infinity, the ice slides over the base with no internal deformation.

The choice above implies that

$$\frac{Ah^{n+1}}{U}[\rho g \sin(\theta)]^n = n+1$$

and thus

$$U = \frac{Ah^{n+1}}{n+1}[\rho g \sin(\theta)]^n$$

which is the dimensional ice velocity at the top surface. The Newtonian case with is n = 1 and A=1/μ gives

$$U = \frac{\rho g h^2 \sin(\theta)}{4\mu} = \frac{g h^2 \sin(\theta)}{4\nu}$$

The non-Newtonian case with n = 3 gives

$$U = \frac{Ah^4}{4}[\rho g \sin(\theta)]^3$$

The case n = ∞ is called "perfect plasticity" because the ice deforms only when the shear stress exceeds a critical value called the "yield stress". The shear stress at the base of the ice is

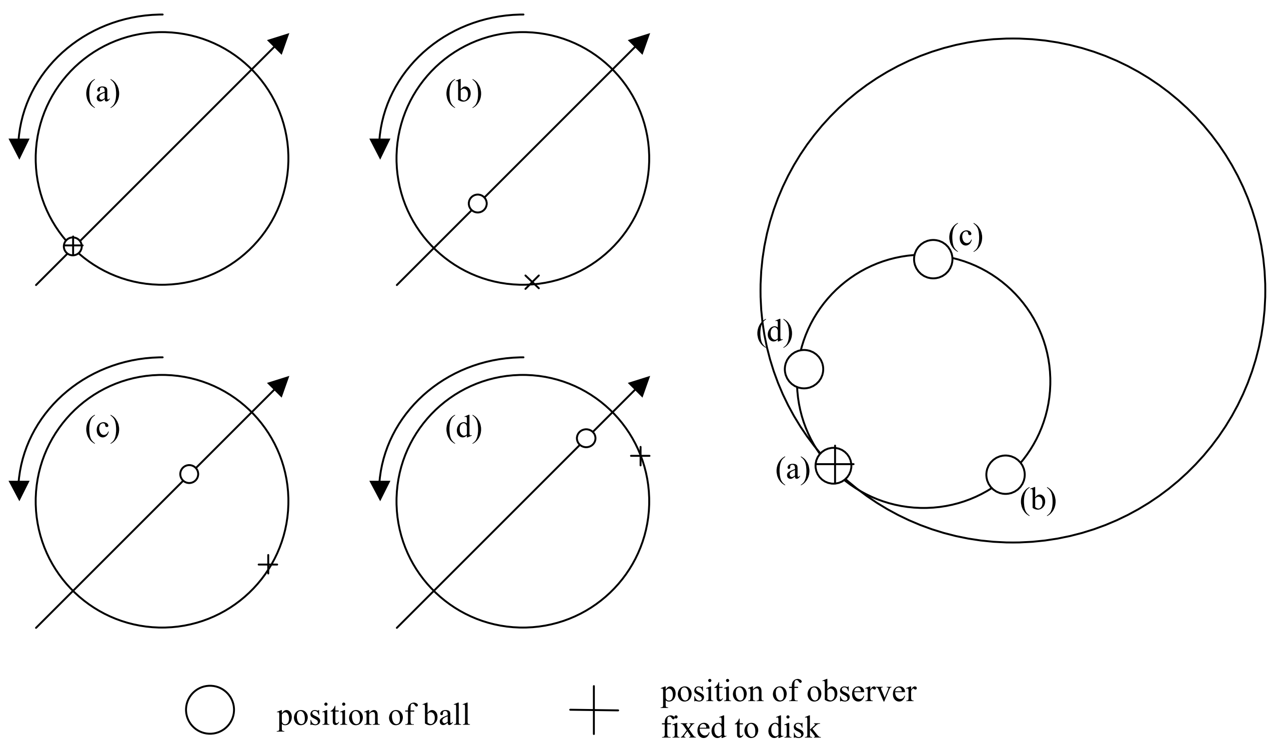
$$\sigma_{base} = \rho g h \sin(\theta)$$

must exceed the yield stress for the ice to move.

Rotating flow

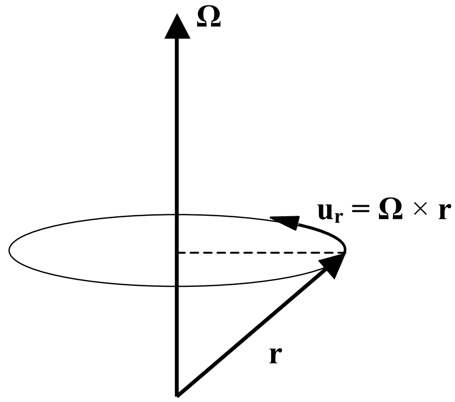
Since Earth rotates and we are forced to rotate with it, we are always making geophysical measurements in an accelerated reference frame. We shall see that this is not always important, but in the oceans and atmosphere, the effects of rotation are critical.

Consider a disk that is rotating about its center (such as a playground merry-go-round). Suppose you throw a ball across the disk along its diameter. If you observe the flight of the ball standing beside the disk you get a completely different impression of the forces acting on the ball than if you are sitting on the edge of the rotating disk.



If you are standing on the ground beside the disk, you see the ball move in a straight line across its diameter. At four successive times its position will be as shown in the four diagrams (a) , (b) , (c) and (d) above. If, however, you are fixed to the disk, you will perceive that the ball curves because the disk turns under the straight path of the ball and your eyes are moving with the disk. In fact, if the disk is rotating fast enough, you will get to the other end of the straight flight of the ball just as it arrives and you can catch it! you will think that the ball has gone in a circle that is half the diameter of the disk even though its actual path was straight along the diameter. If you are unaware that the disk was rotating, you would conclude that the ball is acted on by a mysterious force that causes it to orbit in a circle. This “force” is called the “Coriolis force”. It is not a force at all, but a consequence of trying to explain the motion of a particle using a physical law in a reference frame for which it is invalid. Never-the-less, the observer on the rotating disk gets the right answer if he or she thinks that the Coriolis effect is a real force and so Coriolis force is a useful concept.

We can explicitly derive the Coriolis effect by employing the properties of the cross product. Let $\mathbf{u}'$ be the velocity measured by a non-rotating observer (i.e. with some sort of sky hook) and let $\mathbf{u}$ be the velocity measured by an observer on the surface of the rotating Earth. The two measurements differ by the velocity of rotation $\mathbf{u}_\mathbf{r}$. Thus $\mathbf{u}' = \mathbf{u} + \mathbf{u}_\mathbf{r} = \mathbf{u} + \mathbf{\Omega} \times \mathbf{r}$ as you can see in this figure:



Replacing the velocity with the time derivative of the position vector $\mathbf{r}$

$$\frac{D'}{Dt}\mathbf{r} = \frac{D}{Dt}\mathbf{r} + \mathbf{\Omega} \times \mathbf{r} = \left[\frac{D}{Dt} + \mathbf{\Omega}\times\right]\mathbf{r}$$

From this relation we see that the time derivative operator in the fixed frame is related to the time derivative operator in the rotating frame by

$$\frac{D'}{Dt} = \frac{D}{Dt} + \mathbf{\Omega}\times$$

Note that $\mathbf{r}$ is the same from the point of view of both observers, but the time derivative is not. Now we want to calculate what happens to the acceleration. Assuming that $\mathbf{\Omega}$ is constant

$$\frac{D'\mathbf{u}}{Dt} = \frac{D'}{Dt}\frac{D'}{Dt}\mathbf{r} = \left[\frac{D}{Dt} + \mathbf{\Omega}\times\right]\left[\frac{D}{Dt} + \mathbf{\Omega}\times\right]\mathbf{r}$$

$$= \frac{D^2\mathbf{r}}{Dt^2} + 2\mathbf{\Omega} \times \frac{D\mathbf{r}}{Dt} + \mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{r}$$

$$= \frac{D\mathbf{u}}{Dt} + 2\mathbf{\Omega} \times \mathbf{u} + \mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{r}$$

The term $\mathbf{\Omega} \times \mathbf{\Omega} \times \mathbf{r}$ is the acceleration necessary to keep the rotating observer going around the rotation axis. Because it depends only on position, this "centripetal" term can be moved to the force side of the equation where it behaves just like a gravitational field and can be absorbed into the gravity term. In fact, when gravity is measured at the Earth's surface, you actually measure the total of the force due to the mass attraction and this centrifugal force. This is a major reason that the "International Gravity Formula", which is the standard against which gravity anomalies are judged depends on latitude. The other reason is that the rotating Earth deforms to an ellipsoidal shape so that the distance to the center of mass depends on latitude.

The term $2\mathbf{\Omega} \times \mathbf{u}$ is called the Coriolis term. Because it is perpendicular to **u**, it cannot change the kinetic energy of the mass, only its direction of motion. On the Earth's surface, it causes an apparent force per unit volume $\mathbf{f}_c = -2\rho\mathbf{\Omega} \times \mathbf{u}$ which is always to the right (left) of the velocity in the northern (southern) hemisphere. Its magnitude is $2\Omega|\mathbf{u}|\cos(\theta)$, where θ is the angle from the pole of rotation to the point of observation (called the co-latitude). Thus the Coriolis force is maximum at the pole of rotation and zero at the equator.

If you give a mass such as parcel of water in the ocean a push and it is not acted on by significant viscous forces, it experiences a constant acceleration perpendicular to its velocity. This is precisely the same situation as the mass swinging on a string, the satellite orbiting around a planet or the electron gyrating in a magnetic field. We know that the result must be that the mass travels in a circular path on the Earth's surface. Balancing the magnitude of the Coriolis force and the centrifugal force due to the local curvature gives

$$2\Omega|\mathbf{u}|\cos(\theta) = \eta^2 R$$

where η is the angular frequency of oscillation around the local orbit of radius R. Since $|\mathbf{u}| = \eta R$, we conclude that

$$\eta = 2\Omega\cos(\theta)$$

Like an electron in a magnetic field or a satellite in a gravitational field, the radius of the orbit depends on the initial velocity of the mass. These oscillations are often observed in the ocean. They are called "inertial" oscillations and η is called the "inertial frequency".

The Navier-Stokes Equation

Including the Coriolis term in the momentum equation and absorbing the centrifugal force into gravity, we finally have

$$\frac{D\mathbf{u}}{Dt} + 2\mathbf{\Omega} \times \mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{g} + \nu\nabla^2\mathbf{u}$$

which is often called the Navier-Stokes equation instead of the momentum equation. We have left the Coriolis term on the acceleration side of this equation to emphasize that it is not a force in the usual sense. This equation is still not complete, because fluids such as plasmas have electromagnetic forces in addition to those we have looked at. A full treatment of them is beyond the scope of this course.