11.2.2 Free oscillations

Let us now return to free oscillations. Rewriting (11.16) we get

$$\frac{k\beta}{\sigma} + \frac{\sigma^2 - f^2}{gh} - \left(k^2 + \left(\frac{n\pi}{d}\right)^2\right) = 0. \tag{11.20}$$

Equation 11.20 is cubic in σ. We may anticipate that the three roots will correspond to two gravity waves and a Rossby wave. In general, gravity wave frequencies (for positive h) will exceed f so the term $\frac{k\beta}{\sigma}$ will be negligible. Similarly, for Rossby waves $\sigma^2 \ll f^2$. Thus (11.20) has the following approximate solutions:

$$\sigma^2 \approx gh\left(k^2 + \left(\frac{n\pi}{d}\right)^2\right) + f^2 \tag{11.21}$$

and

$$\frac{k\beta}{\sigma} = \frac{f^2}{gh} + k^2 + \left(\frac{n\pi}{d}\right)^2$$

or

$$\sigma = \frac{k\beta}{k^2 + (\frac{n\pi}{d})^2 + \frac{f^2}{gh}}. \tag{11.22}$$

In contrast to gravity waves, which can have easterly and westerly phase speeds, Rossby waves always have easterly phase speed (relative to U_0). East–west asymmetry is always a characteristic of motions for which β is important.

Several properties of (11.22) are worth noting:

(i) σ has a maximum value for a particular value of k (which you will work out as an exercise). Pedlosky (1979) has an elegant explanation of this which we shall go over as soon as we develop some theorems on vorticity conservation.

(ii) With a constant zonal flow we can replace σ with $\sigma + kU_0$. Also $c = -\frac{\sigma}{k}$. (11.22) becomes

$$c = U_0 - \frac{\beta}{k^2 + (\frac{n\pi}{d})^2 + \frac{f^2}{gh}}. \tag{11.23}$$

Note that as $(k^2 + (\frac{n\pi}{d})^2)$ gets larger the Rossby wave moves more and more nearly with the basic flow U_0. This is completely consistent with observations.

(iii) The velocity fields associated with Rossby waves are almost in geostrophic balance. Nonetheless, the outright assumption of geostrophy would not permit us to evaluate the time evolution of Rossby waves. The development of an appropriate approximate set of equations which exploit geostrophy but still describe Rossby waves will be one of our tasks.

(iv) The dispersive properties of Rossby waves clearly suggests that it is Rossby waves and not gravity waves which describe large-scale meteorological systems.

We next turn to a rather different application of the shallow water equations wherein we again show how β introduces a profound east–west asymmetry.

11.3 Intensification of ocean currents

Figure 11.2 shows a schematic description of the world's ocean currents. A characteristic of these currents is intensification at western boundaries. It is generally accepted that the main current systems in the ocean are forced by wind stresses. However, this does not, *per se*, account for westward intensification. Stommel (1948) first showed that the presence of β (in conjunction with friction in his model) leads to westward intensification. Although Stommel's model is not considered to be an accurate representation of the detailed physics, it is still the simplest model which displays the westward intensification, and we will therefore, concentrate on it. A very general discussion of the wind driven ocean circulations is given in chapter 5 of Pedlosky (1979).

Stommel's equations consist simply in the steady, linearized shallow water equations with vertical turbulent exchange on a β-plane:

$$-fv = \frac{\partial \Phi}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial z}\left(k_v \frac{\partial u}{\partial z}\right) \tag{11.24}$$

$$fu = -\frac{\partial \Phi}{\partial y} + \frac{1}{\rho}\frac{\partial}{\partial z}\left(k_v \frac{\partial v}{\partial z}\right) \tag{11.25}$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{11.26}$$

where ρ is taken as constant. We next multiply (11.24)–(11.26) by ρ and integrate over the depth of the flat-bottomed ocean, H.

$$-fM_y = -H\frac{\partial \Phi}{\partial x} + \int_0^H \frac{\partial}{\partial z}\left(k_v \frac{\partial u}{\partial z}\right) dz \tag{11.27}$$

$$fM_x = -H\frac{\partial \Phi}{\partial y} + \int_0^H \frac{\partial}{\partial z}\left(k_v \frac{\partial v}{\partial z}\right) dz \tag{11.28}$$

and

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} = 0, \tag{11.29}$$

where

$$M_x \equiv \int_0^H \rho u dz$$

$$M_y \equiv \int_0^H \rho v dz.$$

There are contributions to the integrals in (11.27) and (11.28) from both wind stress at the surface and from bottom drag.

$$\int_0^H \frac{\partial}{\partial z}\left(k_v \frac{\partial u}{\partial z}\right) dz = \tau_x - cM_x \tag{11.30}$$

$$\int_0^H \frac{\partial}{\partial z}\left(k_v \frac{\partial v}{\partial z}\right) dz = \tau_y - cM_y, \tag{11.31}$$

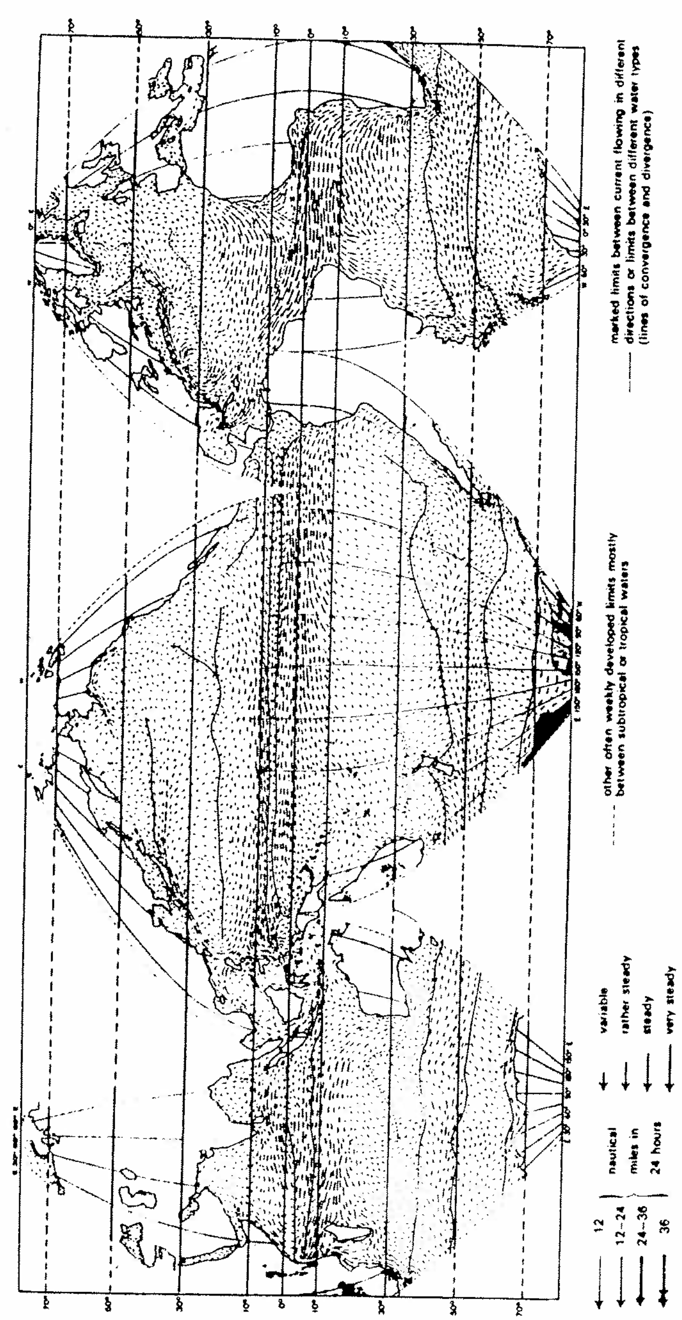

Figure 11.2: The circulation pattern of the world's oceans. This map (after Defant, 1961) represents a long-term compilation of measurements and ship reports. Although it is schematic rather than precise, it reveals the westward intensification of the circulation in the major ocean gyres.

where τ_x and τ_y are wind stress in the x and y directions and c is a bottom drag coefficient. (11.27) and (11.28) become

$$-fM_y = -H\frac{\partial\Phi}{\partial x} + \tau_x - cM_x \qquad (11.32)$$

$$fM_x = -H\frac{\partial\Phi}{\partial y} + \tau_y - cM_Y. \qquad (11.33)$$

To eliminate Φ we differentiate (11.32) with respect to y, differentiate (11.33) with respect to x, and subtract:

$$\begin{aligned} -fM_{y,y} - \beta M_y &= -H\frac{\partial^2\Phi}{\partial x\partial y} + \tau_{x,y} - cM_{x,y} \\ fM_{x,x} &= -H\frac{\partial^2\Phi}{\partial x\partial y} + \tau_{y,x} - cM_{y,x} \\ -\beta M_y &= \tau_{x,y} - \tau_{y,x} - cM_{x,y} + cM_{y,x}. \qquad (11.34) \end{aligned}$$

From (11.29) we can write

$$M_x = -\frac{\partial\psi}{\partial y} \qquad (11.35)$$

$$M_y = \frac{\partial\psi}{\partial x}, \qquad (11.36)$$

where ψ is a stream function. With (11.35) and (11.36), (11.34) becomes

$$-\beta\frac{\partial\psi}{\partial x} = -\nabla\times\vec{\tau} + c\nabla^2\psi. \qquad (11.37)$$

The nature and magnitude of bottom friction are totally uncertain. However, it may be presumed to be very small since the curl of the wind stress is observed to be approximately balanced over the whole interior of the ocean (away from coasts) by the advection of the Earth's vorticity (Welander, 1959); that is, in the interior

$$\beta\frac{\partial\psi}{\partial x} \approx \nabla\times\vec{\tau}. \qquad (11.38)$$

Equation 11.38 is known as the *Sverdrup relation.*

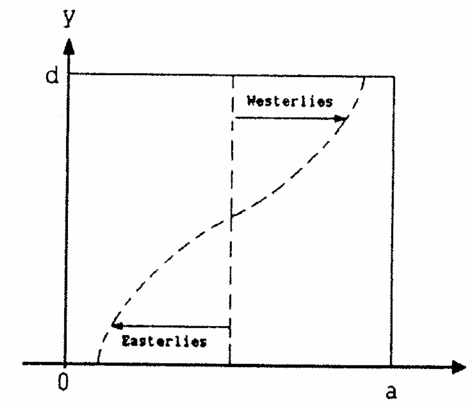

Figure 11.3: Geometry of ocean and wind stress in Stommel model.

For simplicity we will consider our ocean to be rectangular (*viz.* Figure 11.3). Our boundary conditions will simply be

$$\psi = 0 \text{ at } y = 0, d; x = 0, a. \qquad (11.39)$$

(In the absence of lateral friction we can't use no-slip conditions.) We will assume the wind to be purely zonal:

$$\tau_y = 0$$

$$\tau_x = -T\cos\frac{\pi y}{d}. \qquad (11.40)$$

Equation 11.37 becomes

$$-\beta\frac{\partial\psi}{\partial x} = \frac{\pi}{d}T\sin\frac{\pi y}{d} + c\nabla^2\psi. \qquad (11.41)$$

Let

$$\psi = f(x)\sin\frac{\pi y}{d}. \qquad (11.42)$$

Equation 11.41 becomes

$$-\beta f_x = \frac{\pi}{d}T + c\left(f_{xx} - \left(\frac{\pi}{d}\right)^2 f\right). \qquad (11.43)$$

Let us briefly examine the Sverdrup balance

$$-\beta f_x = \frac{\pi}{d} T$$

or

$$f_x = -\frac{\pi}{d}\frac{T}{\beta} \tag{11.44}$$

(i.e., we expect a southward drift everywhere in the interior). Integrating (11.44) we get

$$f = \frac{\pi}{d}\frac{T}{\beta}(-x + k),$$

where k is an integration constant, and

$$\psi_{\text{Sverdrup}} = \frac{\pi}{d}\frac{T}{\beta}(-x + k)\sin\frac{\pi y}{d}. \tag{11.45}$$

Equation 11.45 satisfies boundary conditions at $y = 0$, d, but it can at best satisfy either

$$\psi = 0 \text{ at } x = 0 \ (k = 0)$$

or

$$\psi = 0 \text{ at } x = a \ (k = a).$$

The consequences of either choice are illustrated in Figure 11.4. A situation where an interior solution cannot satisfy a boundary condition generally calls for a thin boundary layer where, although c is very small, cf_{xx} will be able to balance βf_x. Because of β, virtually any added mechanism will cause the return flow to occur on the western boundary. The use of 'bottom friction' illustrates this process. First, let's scale x by $\frac{d}{\pi}$, and f by $\frac{T}{\beta}$. Equation 11.43 becomes

$$-\tilde{f}_{\tilde{x}} = 1 + \epsilon(\tilde{f}_{\tilde{x}\tilde{x}} - \tilde{f}), \tag{11.46}$$

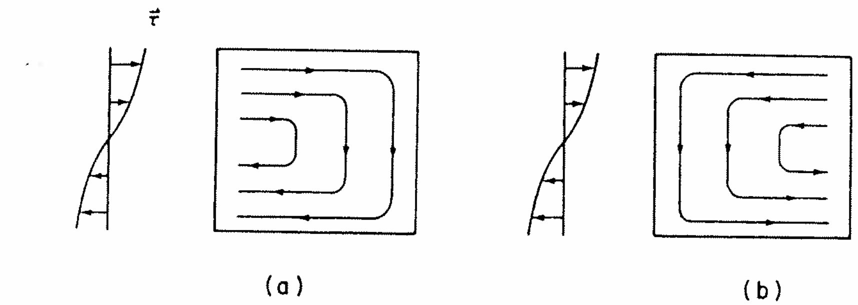

Figure 11.4: (a) The streamlines of the Sverdrup transport for the stress distribution shown at left of the interior geostrophic transport is made to satisfy the zero-normal-flow condition at the eastern boundary. Note the implied requirement of a *western* boundary current to complete the flow. (b) The Sverdrup streamlines for the *same* stress distribution as (a), where now the Sverdrup transport has been made to satisfy the no-normal-flow condition at the western boundary. The validity of this choice implicitly requires the existence of an eastern boundary current. After Pedlosky (1979).

where

$$\begin{aligned} f &= \frac{T}{\beta}\tilde{f} \\ x &= \frac{d}{\pi}\tilde{x} \\ \epsilon &= \frac{c}{\beta}\left(\frac{\pi}{d}\right) \end{aligned}$$

and

$$\tilde{f}(0) = \tilde{f}\left(\frac{\pi a}{d}\right) = 0.$$

Rewrite (11.46)

$$\tilde{f}_{\tilde{x}\tilde{x}} + \frac{1}{\epsilon}\tilde{f}_{\tilde{x}} - \tilde{f} = -\frac{1}{\epsilon}. \tag{11.47}$$

Next let $\tilde{f} = \frac{1}{\epsilon} + g$:

$$g_{\tilde{x}\tilde{x}} + \frac{1}{\epsilon} g_{\tilde{x}} - g = 0. \tag{11.48}$$

Now,

$$g = ae^{\lambda_1 \tilde{x}} + be^{\lambda_2 \tilde{x}},$$

where

$$\begin{aligned} \lambda_{1,2} &= \frac{-\frac{1}{\epsilon} \pm \sqrt{\frac{1}{\epsilon^2} + 4}}{2} \\ \lambda_1 &\approx \epsilon \\ \lambda_2 &\approx \frac{-1}{\epsilon}. \end{aligned}$$

So,

$$\tilde{f} = \frac{1}{\epsilon} + ae^{\epsilon \tilde{x}} + be^{-\tilde{x}/\epsilon} \tag{11.49}$$

$$f(0) = \frac{1}{\epsilon} + a + b = 0 \tag{11.50}$$

and

$$f\left(\frac{\pi a}{d}\right) = \frac{1}{\epsilon} + ae^{\epsilon \pi a/d} + \underbrace{be^{-\pi a/\epsilon d}}_{negligible} = 0. \tag{11.51}$$

Equation 11.51 implies

$$a \approx -\frac{1}{\epsilon} e^{-\epsilon \frac{\pi a}{d}}.$$

Equation 11.50 implies

$$b \approx \frac{1}{\epsilon}(e^{-\epsilon \frac{\pi a}{d}} - 1);$$

and

$$\begin{aligned} \tilde{f} &\approx -\frac{1}{\epsilon}(e^{\epsilon(\tilde{x} - \frac{\pi a}{d})} - 1) + \frac{1}{\epsilon}(e^{-\epsilon \frac{\pi a}{d}} - 1)e^{-\tilde{x}/\epsilon} \\ &\approx -\left(\tilde{x} - \frac{\pi a}{d}\right) - \frac{\pi a}{d} e^{-\tilde{x}/\epsilon}. \end{aligned} \tag{11.52}$$

Thus the interior solution satisfies the eastern boundary condition and the intense return flow occurs at the western boundary. The return flow is confined to a zone of width $\Delta \tilde{x} \sim \epsilon$.

11.4 Remark on Kelvin waves

The case of $v' \equiv 0$

The solutions in Section 2 assume $v' \neq 0$. We here consider the situation where $v' \equiv 0$. There exists a solution in this case known as a Kelvin wave. As an exercise you will derive the properties of Kelvin waves on what is known as an equatorial β -plane (where $f = \beta y$). Here we will take $f = f_0$. Equations 11.1–11.3 become

$$i\sigma u' = -ik\Phi' \tag{11.53}$$

$$fu' = -\frac{\partial \Phi'}{\partial y} \tag{11.54}$$

and

$$iku' = -\frac{i\sigma}{gH}\Phi'. \tag{11.55}$$

From (11.53) and (11.55) we get

$$\frac{\sigma^2}{k^2} = gH, \tag{11.56}$$

while from (11.53) and (11.54) we get

$$\frac{\partial \Phi'}{\partial y} - f_0 \frac{k}{\sigma} \Phi' = 0. \tag{11.57}$$

The solution to (11.57) is

$$\Phi' = Ce^{f_0 (\frac{k}{\sigma}) y}.$$

Note that for $k\sigma < 0$ (i.e., westerly or eastward propagating waves) amplitudes decay away from the boundary at $y = 0$, while for $k/\sigma > 0$ (i.e., easterly or westward propagating waves) amplitudes decay away from the boundary at $y = d$. In a closed basin, Kelvin waves travel around the boundary of the basin in a counterclockwise direction (when $f_0 > 0$). Kelvin waves satisfy the dispersion relation for a gravity wave in the absence of rotation; at the same time, the Kelvin wave's horizontal velocity field is in geostophic balance.