Turbulence

When the Reynolds number becomes sufficiently large, the non-linear term $(\mathbf{u} \cdot \nabla) \mathbf{u}$ in the momentum equation inevitably becomes comparable to other important terms and the flow becomes more complicated. In almost all cases the equation will have more than one solution and at least one of these solutions will be time-dependent and unstable (i.e. will grow with time). Any growing solution will quickly dominate the flow and if there are many of these growing solutions, the flow can become exceedingly complicated (even chaotic). Such a situation is termed "turbulent".

The practical effect of turbulence is to greatly increase the rate at which gradients in momentum (and heat or salt) change. Consider the following measurements of temperature versus depth at Ocean Weather Station (OWS) Papa (50° N, 145 °W) in the northeast Pacific Ocean.



On April 1, the top 120 meters is cold and almost isothermal. By August 1, the top 20 meters has warmed up about 10 degrees due to warming by the spring and summer atmosphere. Two months later on October 1, the thickness of the warm region has roughly doubled to about 50 meters although its average temperature anomaly has decreased to about 8 degrees as the atmospheric heat input fades. The layer again doubles

its thickness in the two months from October 1 to December 1. This implies that the layer thickness is proportional to the square root of time. We have already seen that diffusion predicts exactly this kind of behavior:

$$\delta \approx \sqrt{\kappa \tau}$$

Letting $\tau = 2$ months = 5×10⁶ seconds and $\delta = 20$ meters, we can re-arrange this expression to give

$$\kappa_{effective} \approx \frac{\delta^2}{\tau} = \frac{(20)^2}{5 \times 10^6} \approx 10^{-4} \text{ m}^2/\text{s}$$

as an estimate of the effective diffusivity for this seasonal change in the warm surface layer. This value is approximately 1,000 times the molecular thermal diffusivity of water $(1.5 \times 10^{-7} \text{ m}^2/\text{s})$. This enhanced diffusion is due to turbulent mixing in the upper ocean.

Consider what happens if you start out with a horizontal straight boundary between two regions of uniform temperature and deform it by a rotary motion.



Before the motion starts, molecular diffusion at the interface occurs as we have already discussed and the time scale is the molecular diffusion time. We expect the annual variation in temperature to penetrate only about 2 meters by molecular diffusion. The rotary motion does two things: (1) it advects warm fluid into the cold region and warm fluid into the cold region and (2) it creates thinner and thinner layers that alternate hot and cold. The first process expands the region over which diffusion can be effective

and the second process greatly reduces the length (and hence time) scale over which molecular diffusion needs to act to smooth out the spatial temperature fluctuations. The combination of these two processes results in an apparent diffusivity much higher than the molecular value.

Turbulent fluids have eddies on a spectrum of scales that coexist. Each smaller scale eddy progressively reduces the scale over which molecular diffusion must occur. This cascade to finer and finer structure continues until the molecular diffusion time becomes shorter than the time scale for overturn of the smallest eddies. Then the overall time scale for the process is governed by the time scale for the cascade process and not by the molecular diffusion time scale. This means that "eddy" diffusion of momentum or salt takes the same time as eddy diffusion of heat. The "eddy" diffusivity of a turbulent flow is the same for the diffusion of heat, salt, momentum, etc.

Further insight can be gained by time-averaging the momentum and mass conservation equations. Denote the time average with the symbol $\langle \rangle$. Let

$$\mathbf{u} = \overline{\mathbf{u}}(x, y, z) + \mathbf{u}'(x, y, z, t)$$

with a "mean" flow $\overline{\mathbf{u}} = \langle \mathbf{u} \rangle$ that is independent of time and a fluctuating flow whose time average $\langle \mathbf{u'} \rangle = 0$. Also let the pressure

$$p = \overline{p} + p'$$

with a mean $\overline{p} = \langle p \rangle$ and a fluctuating part with $\langle p' \rangle = 0$. We also assume that the fluid is incompressible and hence $\nabla \cdot \mathbf{u} = 0$.

Because the divergence is a linear operator and $\langle \mathbf{u'} \rangle = 0$

$$\langle \nabla \cdot \mathbf{u'}
angle = \nabla \cdot \langle \mathbf{u'}
angle = 0$$

Then the time-averaged mass conservation equation

$$\langle \nabla \cdot \mathbf{u} \rangle = \langle \mathbf{0} \rangle = \mathbf{0} = \langle \nabla \cdot (\overline{\mathbf{u}} + \mathbf{u'}) \rangle = \nabla \cdot \overline{\mathbf{u}} + \langle \nabla \cdot \mathbf{u'} \rangle$$

implies that

$\nabla \cdot \overline{\mathbf{u}} = 0$

We conclude that the mean flow and the fluctuation separately conserve mass.

The fluctuating parts of all the linear terms in the momentum equation have timeaverages of zero. So the time average of the momentum equation is

$$\rho \left[\frac{\partial \overline{\mathbf{u}}}{\partial \mathbf{t}} + \left\langle (\overline{\mathbf{u}} + \mathbf{u'}) \cdot \nabla (\overline{\mathbf{u}} + \mathbf{u'}) \right\rangle + 2\mathbf{\Omega} \times \overline{\mathbf{u}} \right] = -\nabla \overline{\mathbf{p}} + \rho \mathbf{g} + \nabla \cdot \mu \nabla \overline{\mathbf{u}}$$

where the friction term is written in the form it has if viscosity is not assumed spatially constant. The non-linear term can be expanded as

$$\begin{aligned} \langle (\overline{\mathbf{u}} + \mathbf{u'}) \cdot \nabla (\overline{\mathbf{u}} + \mathbf{u'}) \rangle \\ &= (\langle \overline{\mathbf{u}} \cdot \nabla) \overline{\mathbf{u}} \rangle + \langle (\overline{\mathbf{u}} \cdot \nabla) \mathbf{u'} \rangle + \langle (\mathbf{u'} \cdot \nabla) \overline{\mathbf{u}} \rangle + \langle (\mathbf{u'} \cdot \nabla) \mathbf{u'} \rangle \\ &= (\overline{\mathbf{u}} \cdot \nabla) \overline{\mathbf{u}} + (\overline{\mathbf{u}} \cdot \nabla) \langle \mathbf{u'} \rangle + (\langle \mathbf{u'} \rangle \cdot \nabla \overline{\mathbf{u}} + \langle (\mathbf{u'} \cdot \nabla) \mathbf{u'} \rangle \\ &= (\overline{\mathbf{u}} \cdot \nabla) \overline{\mathbf{u}} + \langle (\mathbf{u'} \cdot \nabla) \mathbf{u'} \rangle \end{aligned}$$

For a general scalar $\,\alpha\,$ and a vector ${\bf A}$

$$\nabla \cdot (\mathbf{A}\alpha) = \alpha \nabla \cdot \mathbf{A} + (\mathbf{A} \cdot \nabla)\alpha$$

Letting $\mathbf{A} = \mathbf{u}'$ and $\alpha = u'$ (the x-component of **u**), we have

$$\nabla \cdot (\mathbf{u}'u') = u'\nabla \cdot \mathbf{u}' + (\mathbf{u}' \cdot \nabla)u' = (\mathbf{u}' \cdot \nabla)u$$

Rearranging this and using the fact that $\nabla \cdot \mathbf{u}' = 0$, the x-component of the fluctuating part of the non-linear term is

$$\langle (\mathbf{u'} \cdot \nabla) u' \rangle = \langle \nabla \cdot (\mathbf{u'} u') \rangle = \nabla \cdot \langle \mathbf{u'} u' \rangle$$

where $\mathbf{u}' = u'\hat{\mathbf{x}} + v'\hat{\mathbf{y}} + w'\hat{\mathbf{z}}$. Likewise, the y-component of the fluctuating part of the nonlinear term is

$$\langle (\mathbf{u}' \cdot \nabla) \mathbf{v}' \rangle = \langle \nabla \cdot (\mathbf{u}' \mathbf{v}') \rangle = \nabla \cdot \langle \mathbf{u}' \mathbf{v}' \rangle$$

and the z-component of the fluctuating part of the non-linear term is

$$\langle (\mathbf{u}' \cdot \nabla) w' \rangle = \langle \nabla \cdot (\mathbf{u}' w) \rangle = \nabla \cdot \langle \mathbf{u}' w' \rangle$$

These equations define the components of a quantity that we will write $\nabla \cdot \langle \mathbf{u'u'} \rangle$. (For those that care, $\mathbf{u'u'}$ is called a "dyadic". This is not something you need to remember. It is the components of its divergence defined just above that are important.)

Finally, the time-averaged momentum equation can be written

$$\rho \left[\frac{\partial \overline{\mathbf{u}}}{\partial \mathbf{t}} + (\overline{\mathbf{u}} \cdot \nabla) \overline{\mathbf{u}} + 2\mathbf{\Omega} \times \overline{\mathbf{u}} \right] = -\nabla \overline{\mathbf{p}} + \rho \mathbf{g} + \nabla \cdot \left[\mu \nabla \overline{\mathbf{u}} - \langle \mathbf{u}' \mathbf{u}' \rangle \right]$$

where the surviving non-linear term due to velocity fluctuations has been moved to the right side of the equation. The two terms inside the square brackets on the right have the units of shear stress. The first is the ordinary shear stress due to molecular viscosity. The

second involves time correlation of fluctuations in the velocity components and is called the "Reynolds" stress. The physical meaning of this term can be understood from this diagram



An eddy must have horizontal velocity in opposite directions at its top and bottom and vertical velocities in opposite directions on opposing sides of the eddy. Thus velocity at points separated by the eddy size must be negatively correlated. We clearly also expect correlation between velocity components perpendicular to one another. The strength of all these correlations as a function of separation gives you information about the spatial spectrum of the eddy sizes and their magnitudes.

The Reynolds stress is a real shear stress and can be measured. So we have made progress, because we now understand that the main consequence of the velocity fluctuations of turbulence is to enhance shear stresses and thus the transport of momentum within the flow. However, it is very difficult to theoretically predict the Reynolds stress in a given situation because it requires a complete solution for the spatial and temporal structure of the velocity fluctuations. In most cases we must resort to empirical results.

The dashed curve in this figure is the parabolic radial variation of non-dimensional laminar velocity versus non-dimensional radius

$$u = 1 - r^2$$

for laminar (stable) flow through a pipe at low Reynolds number (less than about 2,000).



The dotted curve

$$u = \mathbf{A}(1-r)^{\frac{1}{7}}$$

is a widely used fit to experimental measurements of the mean turbulent velocity in a pipe at Reynolds number much larger than 2,000. The scale A has been chosen so that the mean mass flux through the pipe is the same as the laminar case. The main difference between these curves is the flatter variation of the turbulent mean velocity and the center of the pipe and its steeper variation near the wall of the pipe. This suggests that the Reynolds stress can be approximated by

$$-\langle \mathbf{u'u'} \rangle = \mu_{eddy} \nabla \overline{\mathbf{u}}$$

The distortion of the radial velocity structure from its laminar shape requires that the effective "eddy" dynamic viscosity be smaller near the pipe wall and larger in the pipe's center. This is quite reasonable if the efficiency of mixing is governed by the largest

eddy size and the maximum eddy size is governed by the distance from the pipe wall. The solid curve is a recent theoretical result (DeChant, 2005)

$$u = \sqrt{\sin\!\left(\frac{\pi}{2}\sqrt{1-r}\right)}$$

that assumes that turbulent diffusion is a random process governed by the distance from the wall and lends considerable credence to this view of how turbulent diffusion works.

The pressure gradient required to drive a constant mass flux through the pipe when the flow is turbulent is much higher than predicted for molecular viscosity and so eddy dynamic viscosity is everywhere much larger than molecular dynamic viscosity. Assuming that molecular viscosity is small enough to ignore, we can finally approximate the momentum equation for turbulent flow with

$$\frac{\partial \overline{\mathbf{u}}}{\partial \mathbf{t}} + (\overline{\mathbf{u}} \cdot \nabla)\overline{\mathbf{u}} + 2\mathbf{\Omega} \times \overline{\mathbf{u}} = -\frac{1}{\rho}\nabla\overline{\mathbf{p}} + \mathbf{g} + \nabla \cdot (\mathbf{v}_{eddy}\nabla\overline{\mathbf{u}})$$

If V_{eddy} is spatially constant, the "friction" term has the form that we are used to

$$V_{\rm eddv} \nabla^2 \overline{\mathbf{u}}$$

The final conclusion is that mean velocity of a turbulent flow approximately satisfies the same equation as laminar flow, but the effective (eddy) is much larger than its molecular value. When the mean velocity is zero, the momentum equation reduces to the diffusion equation with enhanced diffusivity. We argued above that the diffusivity for other quantities such as heat and salt should be the same as for momentum. Thus the eddy viscosity in the upper ocean at OWS Papa should also be 10^{-4} m²/s.

When stable density stratification is present, vertical motions must overcome the potential energy due to the stratification. If the fluid has insufficient kinetic energy, vertical eddies will be inhibited. Horizontal eddies will not be suppressed. Thus the presence of stable stratification results in anisotropic eddy diffusion. Horizontal turbulent diffusion in the ocean can be expected to be much more efficient than vertical diffusion. Rotation also results in anisotropic turbulent diffusion because motion along the rotation axis is more difficult than motion perpendicular to the axis.