

4. Diffusion

If you ignore all terms in the Navier-Stokes equation except the time-dependent inertia and the friction you get

$$\frac{\partial \mathbf{u}}{\partial t} = \nu \nabla^2 \mathbf{u}$$

You may recognize this as a diffusion equation. Thus viscous effects in fluid mechanics are a diffusive phenomenon. To help us develop intuition about how diffusion works, we shall begin by considering the diffusion of the scalar quantity temperature.

4.1. The thermal diffusion equation

Heat is a concept invented before scientists understood the molecular nature of materials. We now know that it is the volume average of the kinetic energy associated with the vibrational dance that all molecules do at temperatures above absolute zero. This dance is chaotic and when the heat is uniform, net movement averages to zero. On the other hand, if you place a hot object with rapidly dancing molecules next to a cold object with slowly dancing molecules, collisions at the boundary pass momentum from some of the fast molecules to some of the slow ones. The molecules in the cold material near the boundary will then have somewhat higher average energy than they did before and will appear to warm up. The molecules in the hot material near the boundary will have somewhat lower average energy than they did before and will appear to cool down. We say that heat has flowed from the hot material to the cold material. We do not actually measure the kinetic energy of vibration. Instead we measure the effect of the ensemble of vibrating molecules on something else such as the volume of mercury in a thermometer or the electrical resistance of carbon in a thermistor. Historically, these measurements have been reported as a quantity called temperature. There is a particular temperature scale (called the absolute or Kelvin scale) in which the temperature T is directly proportional to the heat (i.e kinetic energy of vibration) per unit volume

$$q = \rho c_p T$$

where c_p is called the specific heat.

The first really quantitative study of heat flow was done by Fourier who conducted a series of careful experiments that in their most general form can be summarized as a relation between the temperature gradient and the directional flux of heat per unit area

$$\mathbf{h} = -\mathbf{K} \cdot \nabla T$$

with components

$$h_x = K_{xx} \frac{\partial T}{\partial x} + K_{xy} \frac{\partial T}{\partial y} + K_{xz} \frac{\partial T}{\partial z}$$

$$h_y = K_{yx} \frac{\partial T}{\partial x} + K_{yy} \frac{\partial T}{\partial y} + K_{yz} \frac{\partial T}{\partial z}$$

$$h_z = K_{zx} \frac{\partial T}{\partial x} + K_{zy} \frac{\partial T}{\partial y} + K_{zz} \frac{\partial T}{\partial z}$$

The thermal conductivity tensor \mathbf{K} is symmetric (i.e. $K_{xy} = K_{yx}$) so it has only six independent elements. These could in principle be predicted using statistical mechanics. More simply, they can be measured. Unless otherwise stated, we shall consider only the case of isotropic diffusion in which the diagonal elements of \mathbf{K} are all equal to K and the off-diagonal elements are zero. Then Fourier's Law reduces to

$$\mathbf{h} = -K \nabla T$$

The conservation of energy implies that the rate of change of heat inside any volume V attached to the material must be equal to the flux of heat into the volume. Thus

$$\frac{D}{Dt} \int_V q \, dV = \int_A -\mathbf{h} \cdot \hat{\mathbf{n}} \, dA$$

where A is the surface of V and $\hat{\mathbf{n}}$ is the outward unit normal. Using the defining relation between heat and temperature on the the left side and applying Gauss' Theorem and using Fourier's (isotropic) Law on the left we obtain

$$\int_V \rho c_p \frac{DT}{Dt} \, dV = \int_V \nabla \cdot \mathbf{h} \, dV = \int_V \nabla \cdot K \nabla T \, dV$$

For simplicity, I have assumed that ρ and c_p are constant (always very good approximations in the ocean) and I will now add the additional simplification that K is constant. Note that this equation is valid for both Centigrade and Kelvin temperatures because only the derivatives of temperature enter and the two scales differ by only a constant. Now the volume used for the above integral is arbitrary as long as it is attached to the material. Therefore the integrals can be equal only if the integrands are equal. Thus we finally have the partial differential equation for isotropic thermal diffusion

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \frac{K}{\rho c_p} \nabla \cdot \nabla T = \kappa \nabla^2 T$$

The quantity $\kappa = \frac{K}{\rho c_p}$ is called the thermal diffusivity. For ocean water its numerical value is $1.4 \times 10^{-7} \, m^2/s$.

The above argument can be extended directly to the diffusion of salinity. The only difference is that the diffusivity for salinity is two orders of magnitude smaller ($1.3 \times 10^{-9} \text{ m}^2/\text{s}$) because the large, heavy salt molecules experience difficulty wandering through the dense forest of water molecules. The derivation can also be extended to the eddy model for turbulent diffusion if T is replaced by its average over time and space \bar{T} and the concept of the transfer of thermal energy by molecular collisions is replaced by the idea that mixing a hot and cold fluid will result in a fluid whose \bar{T} is an average of the two parents. The highly local gradients generated by the mixing process are finally destroyed by molecular processes, but in a much shorter time than predicted by the spatial and temporal scales of \bar{T} . An interesting point is that eddy diffusivity is primarily a property of the dynamics of the flow. This means that salinity, whose molecular diffusivity is much less than the thermal diffusivity, will never-the-less have an eddy diffusivity that is the same as for temperature.

4.2. Estimation

I now want to introduce a technique for figuring out the likely behavior of the solution of a partial or ordinary differential equation without solving it. The basic idea is to say that the solution has characteristic length and time scales over which it has a significant change in value. By “significant” we almost always mean a change in amplitude comparable to the maximum variation of the solution in the region of interest. We approximate each derivative in the differential equation by the range of the solution divided by the appropriate length scale. We then use the approximated differential equation plus prescribed information about some of the scales to deduce information about unknown scales.

We can make this process a little more precise in the following way. We shall consider only one-dimensional diffusion in the vertical direction for which the diffusion equation becomes

$$\frac{\partial T}{\partial t} + w \frac{\partial T}{\partial z} = \kappa \frac{\partial^2 T}{\partial z^2}$$

Substitute the following into this equation

$$\begin{aligned} T &= \Theta T' & w &= W w' \\ t &= \tau t' & z &= \delta z' \end{aligned}$$

where T' , w' , t' , and z' are non-dimensional and the scales Θ , W , τ , and δ are chosen so that the non-dimensional derivatives

$$\frac{\partial T'}{\partial t'} = O(1) \quad \frac{\partial T'}{\partial z'} = O(1) \quad \frac{\partial^2 T'}{\partial z'^2} = O(1)$$

Then the diffusion equation becomes approximately

$$\frac{\Theta}{\tau} + \frac{W\Theta}{\delta} \approx \kappa \frac{\Theta}{\delta^2}$$

Note that Θ can be cancelled out. This means that the amplitude of the temperature enters the solution only as a multiplicative constant and does not affect the shape of the depth variation.

One can justify this idea more rigorously for the many geophysical processes that involve oscillations in space and/or time (i.e. waves) or approximately exponential growth or decay. Then the solutions to the appropriate equations can reasonably be approximated by products of functions like $\sin(\frac{2\pi}{\lambda} x)$, $\cos(\frac{2\pi}{\tau} t)$, $e^{\frac{t}{\tau}}$ or $e^{\frac{z}{\delta}}$, where λ is the wavelength, τ is either the period or a scale time for growth or decay (depending on its sign) and δ is a scale length for growth or decay (often called the “skin” depth). All of these functions have the property that their derivative with respect to space or time is essentially the function itself divided by the appropriate scale ($\frac{\lambda}{2\pi}$, $\frac{\tau}{2\pi}$, τ or δ) and their second derivatives are the function divided by the scale squared. This leads directly to the approximation idea outlined above with the further refinement that we should divide the wavelength or period by 2π before using them as scales whenever we expect wavelike solutions.

Now suppose we want to know how the annual temperature variation diffuses into the ocean. We expect that the variation will decay downwards and thus we must be dealing with an exponential-like and not a wavelike variation in the vertical. Second, the important horizontal length scale is likely to be much larger than the vertical one (we should check whether this is reasonable after the fact). We can therefore ignore the horizontal derivative terms relative to the vertical term and can use the one-dimensional approximate equation above. If we ignore advection (i.e. take $W = 0$) the approximate diffusion equation immediately implies

$$\delta \approx \sqrt{\kappa\tau}$$

where I have renamed the vertical length scale δ , which is the depth (sometimes called the skin depth) at which the annual temperature oscillation will have fallen to about $e^{-1} \approx 40\%$ of its surface magnitude. (It will be down to only 5% of its surface magnitude at a depth of 3δ .) The molecular diffusivity of water is $1.4 \times 10^{-7} \text{ m}^2/\text{s}$. Since the annual temperature cycle is roughly sinusoidal and there are $\pi \times 10^7$ seconds in a year (a handy number to remember along with 10^5 seconds in a day), the appropriate scale time is 0.5×10^7 and $\delta \approx 1$ meter. This scale is small enough that our assumption that it was small compared to any horizontal length scale seems likely to be very good. The diffusivity of rock and soil is very similar to water so this calculation also demonstrates why cellars have a very even temperature all year round.

Another example of implying information from the diffusion equation is the penetration of a temperature change through a 2 km thick ice sheet in Antarctica. Since the Antarctic ice sheet is so cold that there is no free water, conduction theory may be

reasonable. Using the value of κ for water, we get $\tau \approx \frac{L^2}{\kappa} \approx 600,000$ years. The problem with this number is that it is long compared to the typical residence time of about 75,000 years for ice in the sheet. (Residence time is the time from the initial snowfall to the eventual calving of the same ice into the sea at the terminus of the ice sheet.) Thus advection is the dominant process controlling the internal temperature of the sheet and the ice very nearly conserves its temperature as it moves.

From the last calculation, it should be obvious that molecular diffusion would be unlikely to be the dominant process controlling the temperature of the ocean. Figure 14 shows typical vertical profiles of temperature in the arctic, mid-latitudes and the tropics. The steep gradient of temperature from about 100 meters to 1 km in the mid and low-latitudes is called the main or permanent thermocline. The secondary sharp gradient shallower than 100 meters at mid-latitudes is called the seasonal thermocline because it varies from summer to winter. From our earlier calculation it is obvious that molecular diffusion cannot be responsible for this seasonal thermocline. Knowing the depth of the seasonal thermocline, we can calculate an effective vertical eddy diffusivity using

$$\kappa \approx \frac{\delta^2}{\tau}$$

For $\tau = 0.5 \times 10^7$ sec and $\delta = 30$ meters we get $\kappa_{eddy} \approx 2 \times 10^{-4} \text{ m}^2/\text{s}$. We can therefore reasonably argue that mixing processes have increased the diffusivity of the upper ocean by a factor of 100,000. We shall consider several processes that contribute to this mixing later.

The main thermocline does not change with time. Therefore something must balance the effective diffusivity. The only possibility is upward advection. The appropriate form of the approximate diffusion equation must be

$$\frac{W\Theta}{L} \approx \kappa \frac{\Theta}{L^2}$$

which can be solved for an estimate of the advection velocity

$$W \approx \frac{\kappa}{L}$$

Taking the effective diffusivity estimated above for the upper ocean as an upper bound and $L = 1 \text{ km}$ we obtain $W \approx 2 \times 10^{-7} \text{ m/s} \approx 2$ centimeters per day ≈ 6 meters/year as an upper limit on the required vertical advection. This velocity is so small that it cannot be measured directly. We know it is there only because of the existence of the stable main thermocline.

What causes the upward flow? If you look at the vertical temperature profile at high latitudes, you see that it is almost constant. Furthermore the near-surface temperature at high latitudes is the same as the deep temperature in most of the oceans. We can conclude

that water cools and sinks at high latitude and fills up the deep ocean. The circuit is completed by the slow upward flow over the rest of the ocean and by a slight poleward net flow in the surface layers.

4.3. Boundary layers

We now consider a fluid mechanical application that leads to the one dimensional diffusion equation. Suppose we have a horizontal flat plate resting on a fluid half space (see Figure 15). The plate oscillates horizontally and sinusoidally with an angular frequency ω (i.e. it completes one complete cycle in a time $\frac{2\pi}{\omega}$). Like Couette flow, the motion is tangential to the boundary and has no variation in the $\hat{\mathbf{x}}$ or $\hat{\mathbf{y}}$ directions. We are therefore justified in taking $\mathbf{u} = u(z, t)\hat{\mathbf{x}}$, $(\mathbf{u} \cdot \nabla)\mathbf{u} = 0$, $\nabla p' = 0$ and ignoring gravity. The Navier-Stokes equation reduces to

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2}$$

and we see that kinematic viscosity is the diffusivity for momentum. We shall assume that the top surface is at $z = 0$ and the surface boundary condition on u is $u(0, t) = U \cos(\omega t)$. The boundary condition as $z \rightarrow -\infty$ is $u \rightarrow 0$.

This partial differential equation can be solved by assuming $u(z, t) = F(z)G(t)$. We then have

$$\frac{\partial u}{\partial t} = F \frac{dG}{dt} \qquad \frac{\partial^2 u}{\partial z^2} = G \frac{d^2 F}{dz^2}$$

Substituting these in the original equation and dividing by FG leads to the equation

$$\frac{1}{G} \frac{dG}{dt} = \frac{\nu}{F} \frac{d^2 F}{dz^2}$$

The left side of this equation depends only on t , while the right side depends only on z . This can only be true for non-trivial functions G and F if both sides are equal to the same constant. In this problem it is convenient to choose this constant to be $i\omega$. Then

$$\frac{dG}{dt} = i\omega G$$

which has cosines and sines of ωt as solutions and

$$i\omega F = \frac{\nu}{F} \frac{d^2 F}{dz^2}$$

The original partial differential equation has been separated into two ordinary differential equations which are coupled only by the constant. This process is called “separation-of-

variables”.

Assuming that ω is a real number, the solution to the equation for $G(t)$ can be an arbitrary sum of sines and cosines of ωt . However, our surface boundary condition clearly restricts the choice to

$$G = \cos(\omega t) = \text{Re}[e^{i\omega t}]$$

If we try a solution of the form $F = e^{\alpha z}$ in the second ordinary differential equation, we must have

$$i\omega = \nu\alpha^2$$

and hence

$$\alpha = \pm \sqrt{\frac{i\omega}{\nu}} = \pm(1+i) \sqrt{\frac{\omega}{2\nu}}$$

In writing this, I have made use of the fact that $i = e^{\frac{i\pi}{2}}$ and thus

$$\sqrt{i} = e^{\frac{i\pi}{4}} = \cos\left(\frac{i\pi}{4}\right) + i \sin\left(\frac{i\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \frac{(1+i)}{\sqrt{2}}$$

Putting the pieces together, the partial differential equation has the general solution

$$u(z, t) = G(t)F(z) = \text{Re}\left[\left(Ae^{+(1+i)\frac{z}{\delta}} + Be^{-(1+i)\frac{z}{\delta}}\right) e^{i\omega t}\right]$$

where I have let $\delta = \sqrt{\frac{2\nu}{\omega}}$ and the constants A and B are still to be determined.

The term with coefficient B blows up as $z \rightarrow -\infty$, so we must have $B = 0$. The term with coefficient A decays to zero as $z \rightarrow 0$ as required and the boundary condition at $z = 0$ implies that $A = U$. The solution finally becomes

$$u(z, t) = \text{Re}\left[Ue^{\frac{z}{\delta}} e^{i\left(\frac{z}{\delta} + \omega t\right)}\right] = Ue^{\frac{z}{\delta}} \cos\left(\frac{z}{\delta} + \omega t\right)$$

The amplitude of this solution decays by a factor $e^{-1} = 0.37$ at $z = \delta$ and $e^{-3} = 0.050$ at $z = 3\delta$. We see that, except for the factor $\sqrt{2}$ in the definition of δ this behavior is exactly what we would have predicted with the estimation technique described earlier. However, the solution has another feature that we would not have predicted with the approximate analysis; the oscillation at depth has a phase which progressively lags behind that at the surface. The oscillation will be exactly 180° out of phase at $z = \pi\delta$. The plate significantly affects the fluid only out to a distance several times δ .

This region is called a boundary layer and δ the boundary layer thickness. For fixed frequency, the boundary layer becomes thinner as the viscosity drops and in the limit of zero viscosity, which we shall call inviscid, the boundary slips over the fluid without affecting it at all. For fixed viscosity the boundary layer thickness also decreases with increasing frequency. Physically, the information about the direction of the plate has less time before the plate direction reverses and therefore penetrates a smaller distance.

4.4. The Blasius layer

The boundary layer that forms on a semi-infinite plate held parallel to a uniform flow (Figure 16) is called the Blasius boundary layer. Upstream of the plate edge the velocity is everywhere non-zero. If the fluid has non-zero viscosity, however, its velocity must go to zero where it is in contact with the plate. We shall hypothesize the existence of a transition region of thickness $\delta \ll L$ the downstream distance from the edge of the plate. We shall furthermore assume both ρ and the velocity outside the boundary layer (called the main stream) are constant. We can therefore ignore the pressure in the main stream. We shall let the mainstream velocity be $\mathbf{u} = U\hat{\mathbf{x}}$.

The Blasius boundary layer is more complicated than the flows we have considered up to now because it is two-dimensional and has two length scales. In approximating the Navier-Stokes equation, we shall let

$$\frac{\partial}{\partial x} \approx \frac{1}{L} \quad \frac{\partial}{\partial z} \approx \frac{1}{\delta}$$

We obviously want to use the magnitude of the upstream velocity as the horizontal velocity scale because the velocity varies from U outside the boundary layer to zero at the plate. We can deduce the vertical velocity scale by approximating the continuity (conservation of mass) equation

$$\nabla \cdot \mathbf{u} = 0 = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}$$

by

$$\frac{U}{L} \approx \frac{W}{\delta}$$

and therefore

$$W \approx \frac{\delta}{L} U$$

This vertical velocity associated with the deviation of the flow around the slower moving fluid in the boundary layer will be small if the boundary layer is thin.

Since the flow is steady, we can obviously drop the term $\frac{\partial \mathbf{u}}{\partial t}$ in the Navier-Stokes equation and its components become

$$\hat{\mathbf{x}}: \quad u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial x} = \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

and

$$\hat{\mathbf{z}}: \quad u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial z} = \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right)$$

Where p is the non-hydrostatic pressure. Now, since $\delta \ll L$, we have that

$$\frac{\partial^2}{\partial x^2} \ll \frac{\partial^2}{\partial z^2}$$

and we can ignore the derivatives along the plate in the viscous term on the right. Estimating the magnitudes of the non-linear inertia terms on the left (using the scale for W derived above) we get

$$\begin{aligned} u \frac{\partial u}{\partial x} &\approx \frac{U^2}{L} \\ w \frac{\partial u}{\partial z} &\approx \frac{WU}{\delta} \approx \frac{U^2}{L} \\ u \frac{\partial w}{\partial x} &\approx \frac{UW}{L} \approx \frac{U^2 \delta}{L^2} \\ w \frac{\partial w}{\partial z} &\approx \frac{W^2}{\delta} \approx \frac{U^2 \delta}{L^2} \end{aligned}$$

Thus the two non-linear terms in each equation are approximately the same size and we conclude that if wanted to actually solve for the flow, neither could be ignored. We can finally approximate the equations for the two components by

$$\hat{\mathbf{x}}: \quad \frac{U^2}{L} + \frac{\Delta P_x}{\rho L} = \nu \frac{U}{\delta^2}$$

and

$$\hat{\mathbf{z}}: \quad \frac{U^2}{L} \left(\frac{\delta}{L} \right) + \frac{\Delta P_z}{\rho \delta} = \nu \frac{U}{\delta^2} \left(\frac{\delta}{L} \right)$$

where ΔP_x and ΔP_z are typical pressure differences in the horizontal and vertical directions. If we multiply the $\hat{\mathbf{z}}$ equation by $\frac{L}{\delta}$, it is obvious that it is the same as the $\hat{\mathbf{x}}$ equation and that

$$\frac{\Delta P_x}{L} = \frac{L}{\delta} \frac{\Delta P_z}{\delta}$$

or

$$\Delta P_z = \left(\frac{\delta}{L}\right)^2 \Delta P_x$$

Thus ΔP_z is very small compared to horizontal pressure changes and we can conclude that pressure at a point inside the boundary layer is the same as the pressure at a point vertically above it in the main stream. Consequently the horizontal pressure differences inside the boundary layer must be same as in the main stream just outside the boundary layer. In our particular case, the horizontal pressure gradient in the main stream is zero by assumption (it need not be in general) and thus $\Delta P_x = \Delta P_z = 0$ and we can drop the pressure terms. Then either component equation implies that

$$\frac{U^2}{L} \approx \nu \frac{U}{\delta^2}$$

and finally

$$\delta \approx \sqrt{\frac{\nu L}{U}}$$

The Blasius boundary layer thickness increases down stream as the square root of the distance from the edge of the plate (see Figure 16). Since δ grows more slowly than L , it will always be possible to satisfy the requirement $\delta \ll L$ by going far enough down-stream. Alternatively, there will always be a finite region close to the edge of the plate in which the boundary layer solution is invalid. An application which can be understood in terms of the behavior of the Blasius layer is the development of Poiseuille flow down stream from the entrance to a pipe. Blasius layers grow inwards from the walls of the pipe beginning at the entrance (see Figure 17). The region in the center of the pipe outside the layer has not yet felt the influence of the shear stress applied by the wall. Thus Poiseuille flow will occur only after the Blasius layer has grown to completely fill the pipe. This will occur when $L \gg \frac{Ua^2}{\nu}$, where L is the distance down the pipe from the entrance, a is the radius of the pipe and U is the entrance velocity (which can be estimated as the mass flux divided by the cross-sectional area of the pipe).

Tritton (section 11.4) discusses a method to find the actual structure of the Blasius boundary layer called a similarity solution. The key to this technique is to assume that one can write

$$u(x, z) = f(\eta) U$$

where η is a vertical coordinate that stretches as one goes down stream from the edge of the plate. The functional shape f of the velocity variation is the same at each x , but the amount of stretching of the vertical coordinate increases in proportion to the boundary

layer thickness. Thus $\eta = \frac{z}{\delta(x)}$. I will not take time to go over this material in this course because another type of boundary layer that exists when the fluid is rotating is of much greater geophysical interest. One difficulty you may have with his discussion is the fact that Tritton uses the stream function, which I have not yet introduced. The same mathematical technique also applies to the flow when a jet enters a semi-infinite fluid and the details (in Tritton, section 11.6) are easier to grasp and do not employ the stream function. I would suggest reading that section if you want to learn about how the similarity solution works.