8. Instabilities

Up to now we have mostly considered motions that were steady, oscillated or decayed with time. However, the most interesting flows are those that grow spontaneously with time, that is are unstable. We could easily spend an entire academic year on this subject. This chapter is therefore only intended to give a feel for how the understanding of unstable phenomena is approached.

8.1. Thermal convection

We first briefly study one of the most important instablities in geophysics: the overturning motion of a fluid heated from below and cooled from above. It is important in driving plate tectonics, cooling Earth's interior, the formation of clouds and thunderstorms and numerous other phenomena at wide variety of scales. We have already seen that a stratified fluid is stable if and only if density decreases upwards. The reverse of this is not necessarily true. In fact, in the case we consider first, instability requires that the downward decrease in density exceed a finite threshold. Physically, it is necessary that the release of potential energy be finite, because friction always absorbs energy for any finite motion. The release of potential energy must exceed this loss.

8.1.1. Equations for Rayleigh-Benard convection in a horizontal layer

The diffusion equation for temperature

$$\frac{dT}{dt} = \kappa \nabla^2 T$$

was derived in a coordinate system fixed to the mass of the fluid. Thus $\frac{d}{dt}$ needs to be interpreted as the substantial derivative $\frac{D}{Dt}$. In an Eulerian reference frame (fixed to inertial space), we then have

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa \nabla^2 T$$

To this we must ad the momentum and continuity equations. We shall assume the Boussinesq approximation in which the effect of variable density enters only where it multiplies gravity.

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla u + \nabla \frac{p}{\rho_0} = -\frac{\Delta\rho}{\rho_0} g\hat{\mathbf{z}} + \nu \nabla^2 \mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$

where ρ_0 is a typical density within the fluid layer and $\Delta\rho$ is the density deviation from ρ_0

Now consider the horizontal layer shown in Figure 51. The top of the layer is maintained at temperature T_{top} and the bottom at $T_{top} + \Delta T$. The 0th order equations when the fluid is at rest are:

$$\mathbf{u} = 0$$

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2}{\partial z^2}$$

These have a steady-state ($\frac{\partial T}{\partial t} = 0$) solution

$$T_0(z) = T_{top} + \Delta T\left(\frac{d - z}{d}\right)$$

(see Figure 51).

The *coefficient of thermal expansion* is defined by

$$\alpha = \frac{\partial \rho}{\partial T}$$

and we shall assume that α is constant. Then the 0th order density structure is

$$\rho_{00}(z) = \rho_0\left[1 - \alpha(T_0(z) - T_{top)}\right]$$

where, for convenience, we have defined $\rho_0 = \rho(T_{top})$, which is the highest density in the layer (again see Figure 51). If we need it, we can easily find the hydrostatic in the layer by integrating

$$p_0 = p(d) - g\int_d^z \rho_{00}(\xi)d\xi$$

We now consider small perturbations to the 0th order solution. Let:

$$\mathbf{u} = 0 + \mathbf{u}_1$$

$$T = T_0 + T_1 \qquad T_1 \ll T_0$$

$$p = p_0 + p_1 \qquad p_1 \ll p_0$$

$$\frac{\Delta\rho}{\rho_0} = \frac{\rho - \rho_{00}}{\rho_0} = - alphaT_1$$

Substituting these into the thermal, momentum and continuity equations above and then substracting the 0th order equations gives the 1st order equations

$$\frac{\partial T_1}{\partial t} + \mathbf{u}_1 \cdot \nabla T_0 = \kappa\nabla^2 T_1$$

$$\frac{\partial \mathbf{u}_1}{\partial t} + \mathbf{u}_1 \cdot \nabla u_1 + \nabla \frac{p_1}{\rho_0} = - \alpha g T_1\hat{\mathbf{z}} + \nu\nabla^2\mathbf{u}_1$$

$$\nabla \cdot \mathbf{u}_1 = 0$$

where I have deliberately kept the *2nd ord* term $\mathbf{u}_1 \cdot \nabla u_1$ for the moment. Note that

$$\mathbf{u}_1 \cdot \nabla T_0 = w_1 \frac{dT_0}{dz} = -\, w \frac{\Delta T}{d}$$

8.1.2. Non-dimensionalization

The next step is to choose apporpriate scales for each of the variables. There are two obvious scales set by the geometry and temperature boundary conditions:

$$T_1 = \Delta T \theta'$$

$$x, y, z = d(x', y', z')$$

where a $'$ implies a non-dimensional variable and have denoted non-dimensional temperature by θ' so as not to mix it up with ΔT. It is not immediately obvious what to choose as for the other scales, so let

$$t = \tau t'$$

$$p_1 = \Pi p'$$

$$\mathbf{u}_1 = U\mathbf{u}'$$

If we substitue these into the thermal equation

$$\frac{\Delta T}{\tau}\frac{\partial \theta'}{\partial t'} - \frac{U\ \Delta T}{d} w' = \frac{\kappa\ \Delta T}{d^2} \nabla'^2 \theta'$$

Multiplying this equation by $\dfrac{\tau}{\Delta T}$ gives

$$\partial \theta' - \left[\frac{U\tau}{d}\right] w' = \left[\frac{\kappa \tau}{d^2}\right] \nabla'^2 \theta'$$

The terms in [] multiply the advection and diffusion terms. Near the horizontal boundaries, vertical motion will be small and diffusion must dominate vrtical heat flow. Elsewhere in the fluid, advection must be important or the solution would not differ from the 0th order conduction solution. In order for a growing solution to exist, we must also have an important time-dependent term. Thus we expect that *all* terms in this equation must be important. So we choose

$$\left[\frac{U\tau}{d}\right] = 1$$

$$\left[\frac{\kappa \tau}{d^2}\right] = 1$$

These imply $\tau = \dfrac{d^2}{\kappa}$, which is the thermal diffusion time constant for the layer thickness and $U = \frac{= d}{\tau} = \frac{\kappa}{d}$, which is the velocity at which a temperature perturbation diffuses in the layer. Then, dropping the primes

$$\frac{\partial \theta}{\partial t} - w = \nabla^2 \theta$$

is the non-dimensional thermal equation.

The non-dimensional continuity equation is obviously

$$\nabla \cdot \mathbf{u} = 0$$

The momentum equation becomes

$$\frac{U}{\tau}\frac{\partial \mathbf{u}'}{\partial t} + \frac{U^2}{d}\mathbf{u}' \cdot \nabla \mathbf{u}' + \frac{\frac{\Pi}{d}\nabla p'}{\rho_0} = \alpha g \, \Delta T \theta' \hat{\mathbf{z}} + \frac{\nu U}{d^2}\nabla^2 \mathbf{u}'$$

The first term on the right is the bouyant driving force, while the second term is the viscous dissipation. The fundamental physics of convection is a balance between mechanical energy input from thermal bouyancy to mechanical energy dissipation by friction. The friction term must be important if the system is not to run away; so in order to more easily compare the relative importance of the other terms to the dissipation term, we multiply the above equation by $\frac{d^2}{\nu U}$. Then

$$\left[\frac{d^2}{\nu\tau}\right]\frac{\partial \mathbf{u}}{\partial t} + \left[\frac{Ud}{\nu}\right]\mathbf{u} \cdot \nabla \mathbf{u} + \left[\frac{d\Pi}{\nu U \rho_0}\right] = \left[\frac{\alpha g \, \Delta T d^2}{\nu U}\right] + \nabla^2 \mathbf{u}$$

where I have again dropped the primes.

Using our choices for U and τ from our consideration of the thermal equation

$$\frac{d^2}{\nu\tau} = \frac{d^2}{\nu(d^2/\kappa)} = \frac{\kappa}{\nu} = 1/P_r = \frac{1}{Prandtl\ \#}$$

$$\frac{dU}{\nu} = \frac{d\kappa/d}{\nu} = \frac{\kappa}{\nu} = 1/P_r$$

$$\frac{\alpha g \, \Delta T d^2}{\nu U} = \frac{\alpha g \, \Delta T d^3}{\nu\kappa} = R_a = Rayleigh\ \#$$

Now, choose

$$\frac{d\Pi}{\nu U \rho_0} = 1$$

which implies that the scale for the pressure fluctuations due to to thermal bouyancy is

$$\Pi = \frac{\nu\kappa\rho_0}{d^2}$$

and the non-dimensional momentum equation is finally

$$\frac{1}{P_r}\left[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\mathbf{u}\right] + \nabla p = R_a \theta \hat{\mathbf{z}} + \nabla^2 \mathbf{u}$$

You can now see why I have I kept the 2nd order term $\mathbf{u}\cdot\mathbf{u}$. It is of the same size as the 1st order term $\frac{\partial \mathbf{u}}{\partial t}$. Thus when the time derivative term is important, so is the non-linear inertia.

The Prandtl number determines whether the inertia terms are important at all. For example:

Solid Earth	very large ν	$P_r \to \infty$
Oils	moderate to large ν, small κ	$P_r \gg 1$
Water		$P_r = 7$
Stars	very large κ (radiation)	$P_r \to 0$
Mercury	very large κ (metallic conduction))	$P_r \ll 1$

Thus all situations are possible in nature and can be realized experimentally. We are only going to consider the case $P_r \to \infty$. Then the complete system of non-dimensional equations is

$$\nabla p = R_a \theta \hat{\mathbf{z}} + \nabla^2 \mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\partial \theta - w = \nabla^2 \theta$$

Note that infinite Prandtl number does not completely remove time-dependence even though the time-dependent inertia is neglected. It still survives in the thermal equation.

8.1.3. Onset of convection

We now want to ask whether there is a *marginally stable* solution to the above system. By this we mean a solution that neither grows nor decays (i.e. is right on the boundary between growing and decaying solutions). An important question is whether this marginally stable state *oscillates*. It can be proven (although I will not) that in this case the marginally stable state does not oscillate. Thus it must be time-independent with $\frac{\partial \theta}{\partial t} = 0$.

For simplicity, consider only 2-D motion in the x-z plane. We then get four equations for the four unknowns p, u, w and θ

$$\frac{\partial p}{\partial x} = \nabla^2 u$$

$$\frac{\partial p}{\partial z} = R_a \theta + \nabla^2 u$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$-w = \nabla^2 \theta$$

Elimating p, u and w by cross-differentiation and substitution leads to an equation for θ alone

$$\nabla^{\nabla^2} \nabla^2 \theta = R_a \frac{\partial^2 \theta}{\partial x^2} = 0$$

The presence of the Laplacian operator ∇^2 suggests that we should try a sinusoidal solution (which could be formally deduced by separation of variables). Since T is fixed at the top and bottom of the layer, the boundary condtion on its perturbation is $\theta = 0$ on the horizontal boundaries. If we further assume that the layer extends to infinity in the horizontal, we expect an infinite of the solution in the horizontal direction. So we can try

$$\theta = \sin kx \sin \pi z$$

where k is the horizontal wavenumber (see Figure 52(a)). Then

$$\nabla^2 \rightarrow -(\pi^2 + k^{2)}$$

$$\frac{\partial^2}{\partial x^2} \rightarrow -k^2$$

and the partial differential equation for θ becomes an algebraic one

$$-(\pi^2 + k^2)^3 + R_a = 0$$

which can be immediately solved for R_a. Thus R_a is not arbitrary. For each chice of the horizontal wavenumber k, only one value of R_a is allowed. For other values of R_a, the time derivative will not be zero. It should be self-evident that R_a larger than the "critical" value must have a growing solution because the bouyancy forcing is stronger than required for marginal stability. Conversely, R_a smaller than the "critical" value must have a decaying solution because there is insufficient bouyancy forcing to overcome friction.

If we plot R_a against k (Figure 52(b)) we get a concave upward curve. There is an absolute minimum value of R_a, which we call the critical Rayleigh number, R_{ac} below which no motion is possible and above which we expect motion. If R_a (i.e. ΔT) is slowly increased from 0, we would expect to see 2-D convection start at R_{ac} and have the corresponding wavenumber k_c. To find R_{ac} and k_c, we let $\frac{\partial R_a}{\partial k} = 0$. Since

$$k^2 R_a = (\pi^2 + k^2)^3$$

differentiation by k gives

$$2kR_a + k^2 \frac{\partial R_a}{\partial k} = 3(\pi^2 + k^2)^2 2k$$

The second term on the left is 0, so we must have

$$R_a = 3(\pi^2 + k^2)^2$$

But we already have

$$R_a = \frac{(\pi^2 + k^2)^3}{k^2}$$

Setting these two expressions equal to each other, we can easily find that

$$k_c = \frac{\pi}{\sqrt{2}}$$

which implies a non-dimensional wavelength of $\sqrt{2}$ or a dimensional wavelength $\sqrt{2}$ times the depth of the layer.

Substituting k_c back into one of the expressions for R_a we have

$$R_{ac} = \frac{27\pi^4}{2} = 657.5 = \frac{\alpha g \Delta T d^3}{\kappa \nu}$$

There is an important caveat to the above calculation. The solution

$$\theta = \sin kx \sin \pi z$$

leads to the conclusion that

$$u \ \mathit{proportion} \cos kx \cos \pi z$$

Thus the horizontal velocity is maximum at the top and bottom of the layer. Thus this solution is only compatible with stress-free horizontal boundaries. Solutions with one or two no-slip boundaries alos exist, but are more complicated. R_{ac} inceases as the number of no-slip boundaries increases. It is 1708 for two.

Note also that

$$\theta = \sin kx \sin n\pi z$$

with n odd are also solutions. However, these higher modes all have higher values of R_{ac} and would not be observed because the 1st mode starts first and once convenction starts, the marginally stable analysis breaks down. However, whatever does happen after the onset of convection, its temperature structure will be strongly prejudiced by that which first exists.

8.2. Double Diffusion

When density depends on two quantities that *diffuse at different rates*, an ostensible stable situation can become uinstable. For instance, heat diffuses 100 times faster than salt in the ocean. Figure 53(a) sketches a T-S plot for an initially stable situation with cold, relatively fresh water (1) beneath warm, salty, but less dense water (2). If a parcel of

(1) is raised into (2), it will warm up long before there is much change in its salinity and its density (3) will become *less* than (2). The parcel will continue to rise. Since this phenomenon is more effective if the scale on which the diffusion occurs is small, what is actually observed in a fluid is horizontally thin "plumes" called salt fingers (see Figure 53(b)).

This precise situation occurs when Antarctic Bottom Water flows under North Atlantic Deep (NAD) water and at the top of the NAD at lower latitudes, where surface water is warmed by the sun and made salty by evaporation. The resultant "double diffusive" convection is a significant contributor to vertical mixing in these two situations. It is has been suggested that one could use oceanic stratification to generate energy by using a vertical tube extending through the thermocline. This tube would allow heat to diffuse and warm the water as it rose in the pipe. The salinity difference between the surface and deep water could generate about a 1 meter pressure head that could drive an electric generator.